RANK-FINITENESS AND THE NATURE OF THE UNIVERSE

PAUL BRUILLARD, SIU-HUNG NG, ERIC C. ROWELL, AND ZHENGHAN WANG

1. Introduction

One hundred and fifty years ago Mendeleev published [4] the first version of our modern periodic table. He ordered the sixty-three known elements by atomic mass and organized them into rows and columns, leaving gaps for as-yet-to-be-discovered elements (see Figure 1).

This first attempt at classifying and categorizing the basic building blocks of our universe inspired generations of scientists and is one of the major scientific accomplishments of the modern era.[1] These days we order the one hundred and eighteen known elements by the number of protons in their nucleus which uniquely determines them, and is closely related to their atomic mass. We also organize them in rows (periods) and columns (groups) corresponding to the number of electron shells and by chemical properties, much like Mendeleev's table. Although it is not known if there are more that one hundred and eighteen elements, the periodic table of 2019 does have a refreshingly complete look to it!

More than two millennia before Mendeleev, Plato and Aristotle had been thinking about the elements too–they postulated five: fire, earth, air, water and æther. This last element was a bit speculative, but they provided a compelling argument for its existence: there are five platonic solids: the tetrahedron, cube, octahedron, icosahedron and dodecahedron. Although they were somewhat naive on a few points, we can find inspiration in these ancient Greeks' ideas: symmetry can be very useful for classification! Indeed, periodic tables these days often include symmetry information on the crystal structure of the elements when they organize themselves into molecules. Diamond (everyone's favorite allotrope of carbon) has 48 symmetries that fix a point. More generally, all solid crystals can be categorized by their space groups, of which there are exactly 230. We can also appreciate the Platonic approach to classification: reduce a scientific question (*how many elements are there?*) to a geometric question (*how many regular convex polyhedra are there?*) which then can be answered using algebra/number theory (*Ask Theaetetus!*[2]).

Date: July 8, 2019.

[1]The United Nations General Assembly and UNESCO designated 2019 as the International Year of the Periodic Table of Chemical Elements.

[2]Probably he used the fact that the sum of the angles at a vertex consisting of k n-gons must satisfy $\frac{n-2}{n}k\pi < 2\pi$, as is found in Euclid's Elements book XIII.

ОПЫТЪ СИСТЕМЫ ЭЛЕМЕНТОВЪ,

ОСНОВАННОЙ НА ИХЪ АТОМНОМЪ ВѢСѢ И ХИМИЧЕСКОМЪ СХОДСТВѢ.

			Ti=50	Zr= 90	?=180.
			V=51	Nb= 94	Ta=182.
			Cr=52	Mo= 96	W=186.
			Mn=55	Rh=104,4	Pt=197,1.
			Fe=56	Ru=104,4	Ir=198.
			Ni=Co=59	Pd=106,6	Os=199.
H=1			Cu=63,4	Ag=108	Hg=200.
	Be= 9,4	Mg=24	Zn=65,2	Cd=112	
	B=11	Al=27,3	?=68	Ur=116	Au=197?
	C=12	Si=28	?=70	Sn=118	
	N=14	P=31	As=75	Sb=122	Bi=210?
	O=16	S=32	Se=79,4	Te=128?	
	F=19	Cl=35,5	Br=80	I=127	
Li=7	Na=23	K=39	Rb=85,4	Cs=133	Tl=204.
		Ca=40	Sr=87,6	Ba=137	Pb=207.
		?=45	Ce=92		
		?Er=56	La=94		
		?Yt=60	Di=95		
		?In=75,6	Th=118?		

Д. Менделѣевъ

FIGURE 1. Mendeleev's 1869 Periodic Table

Quantum mechanics allows for a more detailed classification of the matter that make up our universe. These elementary particles, e.g., quarks, photons, neutrinos etc., come in two different types: fermions and bosons. They are distinguished by a different kind of symmetry called *spin statistics*. This refers to their behavior when two identical particle types switch places: the wave function for fermions acquires a sign, while the wave function of two bosons is unchanged. Fierz and Pauli showed that for point-like particles in 3 spacial dimensions only bosons and fermions are possible. Mathematically, this is manifested in the triviality of the fundamental group of $\mathbb{R}^3$ with a finite set of points deleted.

What can happen in 2 spacial dimensions? This may seem like an academic question–after all, our universe has 3 spacial dimensions. However, since the 1970s the study of physical systems at near absolute zero allow for states of matter, in which the particles are effectively confined to 2 dimensions due to energy constraints. In 2016 the Nobel Prize for Physics was awarded to Thouless, Kosterlitz and Haldane for their pioneering work on *topological phases of matter* (TPMs) which could support (quasi-)particles that are neither bosons or fermions, but rather *anyons*. The exchange of two identical anyons can introduce a scale factor $\theta = e^{2\pi i r}$ to the wave function, where r can be *any* rational number $r \in \mathbb{Q}$. Mathematically, there are no constraints on how many different types of anyons that can exist, in contrast to the 3 dimensional boson/fermion dichotomy. Indeed, the fundamental group of a disk with n points removed is as big as possible: the free group on n symbols. If 3 ore more anyons are present in an anyonic system then it is even possible that the wave function lives in a high-dimensional vector space, and could have non-abelian braiding statistics! The possibility of building quantum computers that were topologically protected against decoherence based on anyons was initiated by Freedman and Kitaev in the late 90s and is currently being explored.

How can we classify topological phases of matter? How many are there? What are some ways of distinguishing them from each other? Is there a "periodic table" of topological phases of matter? Around 2003 it was realized that very promising mathematical models for topological phases are *modular tensor categories* (MTCs). These had been studied for a decade or so, going back to the work of Moore and Seiberg in Conformal Field Theory and Turaev's approach to Quantum Topology in the early 1990s. These are elegant algebraic structures consisting of a finite number of isomorphism classes of simple objects and maps between them that amazingly interact in a way similar to those expected of anyons. For example, the tensor product of objects X and Y corresponds to the *fusion* of anyons, denoted $X \otimes Y$, and the direct sum $X \oplus Y$ is related to the superposition principle. Crucially, an MTC comes with braiding maps $c_{X,Y} : X \otimes Y \to Y \otimes X$, that describe the braiding statistics. The possibility of anyons existing in topologically more interesting surfaces, e.g. a torus, can also be modeled by the rich structure of MTCs. The relationship between MTCs and topological phases of matter is so close that a pocket dictionary (see [7, Table 1]) can be made to facilitate discussions between mathematicians and condensed matter physicists. For example, the "neutral" or trivial anyon type representing the vacuum corresponds to the tensor unit object $\mathbf{1}$, and the anti-particle type of X correspond to the dual object X^*.

Each anyonic system harbors a finite number of distinguishable anyon types. A given configuration may have an arbitrary number of such anyons, but their types must be chosen from among a fixed finite collection (including the vacuum). This is encoded in the *rank* r of the modeling MTC $\mathcal{C}$, i.e., the number of (isomorphism classes of) simple objects. Rank seems like a useful parameter for labeling the rows of a hypothetical periodic table of TPMs, and in 2003 Wang suggested that each such row would have finite length–the Rank-Finiteness conjecture. The evidence from 3 dimensions where only bosons and fermions are possible could easily mislead us: lowering dimension can play havoc on your intuition! Indeed, we have already seen through the fundamental group that points in 3 dimensions are much less complicated than points in 2 dimensions. Notice also that while there are 5 regular convex polyhedra, there are infinitely many regular convex polygons!

The authors met for the first time as a group at the AIM Workshop "Classifying Fusion Categories" and began a joint project with the modest goal of classifying rank 5 MTCs. A few months in it was realized that a proof of the rank-finiteness conjecture was within reach! The idea of the proof goes as follows:

Having successfully reduced the scientific question (*Are there finitely many TPMs with a given fixed number of anyon types?"*) to a mathematical conjecture, we need to reduce it further to a tractable number theoretical question. One approach is to consider the dimensions d_X of the simple objects X. These are not ordinary integer dimensions $d \in \mathbb{Z}$ as a vector space has–instead they are (real) *algebraic* integers: numbers x satisfying a monic polynomial $x^n + a_{n-1}x^{n-1} + \cdots + a_0$ with the $a_i \in \mathbb{Z}$. Actually they are slightly nicer *cyclotomic* integers that can be expressed as sums and products of roots of unity such as $e^{2\pi i/\ell}$. The d_X behave in a way that justifies calling them dimensions: they are multiplicative with respect to $\otimes$ and additive with respect to $\oplus$, and are *asymptotically* equal to the dimension of the state space of n anyons in a disk. We can define the global dimension of the rank r MTC $\mathcal{C}$ by the equation:

$$\dim(\mathcal{C}) = 1 + d_1^2 + \cdots d_{r-1}^2 \tag{1}$$

where the d_i are the dimensions of the distinct isomorphism classes of simple objects in $\mathcal{C}$. This equation is inspired by finite groups: for a finite group G one can prove that $|G| = \sum_{V \in \mathrm{Irr}(G)} (\dim(V))^2$, where $\mathrm{Irr}(G)$ is the set of (isomorphism classes of) irreducible matrix representations over the complex numbers $\mathbb{C}$. Eq. (1) plays a starring role in the proof of rank-finiteness. A result known as *Ocneanu rigidity* (see [2]) can be used to reduce the question of rank-finiteness to showing that eq. (1) has finitely many solutions when the d_i and and $\dim(\mathcal{C})$ correspond to a rank r MTC.

Eq. (1) is a generalized version of a *Diophantine* equation: a polynomial equation whose integer-valued solutions are sought. A famous family of Diophantine equations is $a^n + b^n = c^n$ for $n \geq 2$. For $n = 2$ there are infinitely many solutions, whereas Wiles' Theorem tells us that for $n \geq 3$ there are no solutions. In our case we are looking for solutions in the larger ring of algebraic integers. Without using further properties of MTCs we are immediately stuck: $Y = 1 + X_1^2 + \cdots + X_{r-1}^2$ clearly has infinitely many integer solutions, not to mention the algebraic integer solutions! Let us take a look at the old problem of classifying primitive Pythagorean triples: $(a, b, c) \in \mathbb{Z}^3$ where $a^2 + b^2 = c^2$ where a, b and c have no common divisors. Euclid showed that all primitive solutions are of the form $(m^2 - n^2, 2mn, m^2 + n^2)$ where m and n share no common divisors and are not both odd, up to switching a and b. What if we added a hypothesis that any prime dividing a, b or c must come from a finite collection of primes $\{p_1, \ldots, p_k\}$? It is a highly non-trivial fact that, under this restriction, there are only finitely many solutions! This is a highly non-trivial result in analytic number theory, which has been generalized to algebraic integers [3].

This reduces the problem of proving rank-finiteness to showing that, for a fixed r, the set of primes that can divide the algebraic integer $\dim(\mathcal{C})$ coming from all rank r MTCs $\mathcal{C}$ is finite. Here "prime" and "divides" are interpreted in the ring theory sense of ideals. To achieve this we first bound the *Frobenius-Schur exponent* $\mathrm{FSexp}(\mathcal{C})$ in terms of the rank r:

this is an integer invariant akin to the exponent of a finite group, i.e., the LCM of the orders of all of the elements of G. Bounding $\mathrm{FSexp}(\mathcal{C})$ is an application of Galois theory, found in [5], and it follows that there are only finitely many primes dividing $\mathrm{FSexp}(\mathcal{C})$. Finally, one shows that the set of primes dividing $\dim(\mathcal{C})$ coincides with the set of primes dividing $\mathrm{FSexp}(\mathcal{C})$: this is a categorical version of Lagrange's and Cauchy's theorems from group theory: the set of primes dividing $|G|$ is equal to the set of primes appearing as orders of elements of G.

Rank-finiteness for MTCs is yet another example of our infinite universe having a beautifully ordered nature when viewed in the light of mathematics. From another point of view, our universe inspires the development of deep mathematics.

References

[1] P. Bruillard, S.-H. Ng, E. Rowell, and Z. Wang. *Rank-finiteness for modular categories.* J. Amer. Math. Soc. **29** (2016), no. 3, 857-881.

[2] P. Etingof, D. Nikshych and V. Ostrik, *On fusion categories*, Ann. of Math., 162 (2005), no. 2, 581–642.

[3] J.-H. Evertse, On Sums of S-Units and Linear Recurrences, Compositio Mathematica (1984), no. 53, 225244.

[4] D. Mendelejeff, *Über die Beziehungen der Eigenschaften zu den Atomgewichten der Elemente,* Zeitschrift fr Chemie 12, 405-406 (1869);

[5] S.-H. Ng, and P. Schauenburg. *Frobenius-Schur indicators and exponents of spherical categories.* Adv. Math. **211** (2007), no. 1, 34–71.

[6] E. Rowell, R. Stong, and Z. Wang. *On Classification of Modular Tensor Categories.* Comm. Math. Phys. **292** (2009), no. 2, 343–389.

[7] E. C. Rowell and Z. Wang. *Mathematics of topological quantum computing*, Bull. Amer. Math. Soc. 55 (2018), 183-238

E-mail address: pbruillard@expedia.com

Expedia Group, Bellevue, WA, U.S.A

E-mail address: rng@math.lsu.edu

Department of Mathematics, Louisiana State University, Baton Rouge, LA U.S.A.

E-mail address: rowell@math.tamu.edu

Department of Mathematics, Texas A&M University, College Station, TX U.S.A.

E-mail address: zhenghwa@microsoft.com

Microsoft Research Station Q and Department of Mathematics, University of California, Santa Barbara, CA U.S.A.