Some Analogues of Circulant Matrices

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Circulant matrices of size n are those whose i, j entry is a function of $(i-j) \bmod n$. That is, there are n numbers $c_0, c_1, \ldots, c_{n-1}$ and the i, j entry of the matrix is $c_{(i-j) \bmod n}$. Each row is a cyclic permutation of the row above and likewise for the columns–hence the name "circulant." (Toeplitz matrices have the form $[c_{i-j}]$ and Hankel matrices have the form $[c_{i+j}]$.)

Now modify the previous definition to use the operation of "dyadic" subtraction rather than ordinary subtraction. This gives a new class of matrices with interesting properties. Dyadic addition is defined for any two natural numbers i and j by adding their binary representations bit by bit without carries. For example, $5 = 101_2$ and $7 = 111_2$. Adding them without carries gives $010_2 = 2$. Thus the dyadic sum of 5 and 7 is 2. We denote this operation by $\oplus$, not to be confused with the direct sum of algebraic objects in other contexts. Dyadic addition makes the natural numbers into an abelian group with 0 as the identity element and each element as its own inverse. The numbers from 0 to $2^n - 1$ form a subgroup isomorphic to the n-fold product of $\mathbf{Z}_2$. Since addition and subtraction are the same here, we can use them interchangeably.

Define a square matrix to be a "Walsh" matrix if its i, j entry is a function of the dyadic sum $i \oplus j$. The name is chosen because of a connection with the system of Walsh functions. An 8 by 8 Walsh matrix has this form:

$$\begin{bmatrix} a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 \\ a_1 & a_0 & a_3 & a_2 & a_5 & a_4 & a_7 & a_6 \\ a_2 & a_3 & a_0 & a_1 & a_6 & a_7 & a_4 & a_5 \\ a_3 & a_2 & a_1 & a_0 & a_7 & a_6 & a_5 & a_4 \\ a_4 & a_5 & a_6 & a_7 & a_0 & a_1 & a_2 & a_3 \\ a_5 & a_4 & a_7 & a_6 & a_1 & a_0 & a_3 & a_2 \\ a_6 & a_7 & a_4 & a_5 & a_2 & a_3 & a_0 & a_1 \\ a_7 & a_6 & a_5 & a_4 & a_3 & a_2 & a_1 & a_0 \end{bmatrix} \tag{1}$$

A Walsh matrix of size 2^n is defined by 2^n numbers a_k, for $0 \le k \le 2^n - 1$. If the size is

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not a power of 2, say m where $2^{n-1} < m < 2^n$, then there are still 2^n independent entries a_k, $0 \le k \le 2^n - 1$. We will only consider Walsh matrices of size 2^n. For them there is an alternative description.

Define $\mathcal{W}$ to be the smallest set of square matrices satisfying these properties.

W1. $\mathcal{W}$ contains all 1 by 1 matrices.

W2. If $A, B \in \mathcal{W}$, then $\begin{bmatrix} A & B \\ B & A \end{bmatrix} \in \mathcal{W}$.

Theorem 1. *$\mathcal{W}$ is the set of Walsh matrices whose sizes are powers of 2.*

Proof. First, suppose that $C = [c_{i \oplus j}]$ is a Walsh matrix of size 2^n. Break C into 2 by 2 block form with submatrices of size 2^{n-1}. The upper left corner is identical to the lower right corner, because $i \oplus j = (i + 2^{n-1}) \oplus (j + 2^{n-1})$. Similarly, the lower left corner is identical to the upper right corner, because for $2^{n-1} \le i \le 2^n - 1$ and $0 \le j \le 2^{n-1} - 1$, we see that $i \oplus j = (i - 2^{n-1}) \oplus (j + 2^{n-1})$. Thus, C has the correct block form, and we can proceed recursively on the smaller submatrices.

For the reverse inclusion we use induction. Clearly, all 1 by 1 matrices in $\mathcal{W}$ are Walsh matrices. Suppose that the matrices of size less than 2^n in $\mathcal{W}$ are Walsh matrices and consider a matrix of size 2^n

$$C = \begin{bmatrix} A & B \\ B & A \end{bmatrix} \in \mathcal{W}.$$

Thus, for $0 \le i, j \le 2^{n-1} - 1$

$$c_{ij} = c_{i+2^{n-1},j} = a_{ij}$$

$$c_{i+2^{n-1},j} = c_{i,j+2^{n-1}} = b_{ij}$$

from which it is straightforward to check that c_{ij} is a function of $i \oplus j$. $\square$

Using the recursive block characterization of Walsh matrices enables us to quickly determine the eigenvalues and eigenvectors. First we note that if A and B are Walsh matrices of the same size, then $AB = BA$. The proof is easy by induction and is left to the reader.

The fundamental result needed to compute the eigenvalues and eigenvectors is the next theorem.

Theorem 2. *If A and B are commuting matrices and v is an eigenvector of both A and B with eigenvalues λ and μ, respectively, then (v, v) and $(v, -v)$ are eigenvectors of $\begin{bmatrix} A & B \\ B & A \end{bmatrix}$ with eigenvalues $\lambda + \mu$ and $\lambda - \mu$, respectively.*

Proof. Left to the reader. $\square$

Consider the 4 by 4 Walsh matrix

$$\begin{bmatrix} a_0 & a_1 & a_2 & a_3 \\ a_1 & a_0 & a_3 & a_2 \\ a_2 & a_3 & a_0 & a_1 \\ a_3 & a_2 & a_1 & a_0 \end{bmatrix}$$

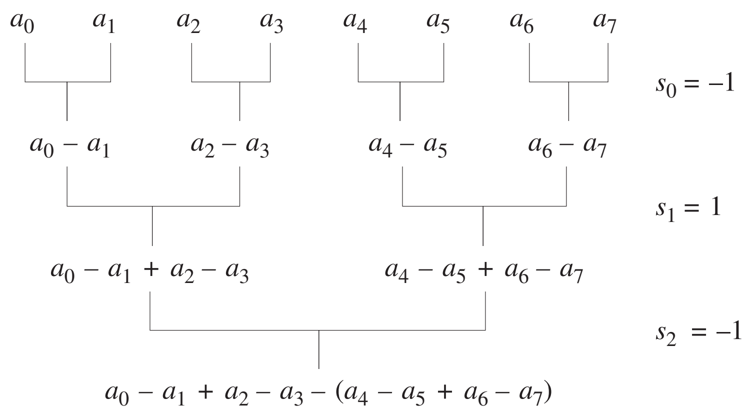


Figure 1: Constructing an eigenvalue.

The eigenvalues of the 2 by 2 submatrices are $a_0 + a_1$, $a_0 - a_1$ and $a_2 + a_3$, $a_2 - a_3$. The sums go with eigenvector $(1, 1)$ and the differences go with $(1, -1)$. Therefore, letting $v = (1, 1)$, we get the following eigenvectors with associated eigenvalues

$$\begin{array}{cc}
(1,1,1,1) & a_0 + a_1 + a_2 + a_3 \\
(1,1,-1,-1) & a_0 + a_1 - a_2 - a_3) \\
(1,-1,1,-1) & a_0 - a_1 + a_2 - a_3 \\
(1,-1,-1,1) & a_0 - a_1 - a_2 + a_3)
\end{array}$$

The construction of the eigenvalues and eigenvectors from the coefficients a_i can be visualized by constructing a complete binary tree of depth n with the 2^n coefficients $a_0, \ldots, a_{2^n-1}$ at the leaves (at the top). Considering the branches at the top as level 0, label the right branches of level i with a sign $s_i = \pm 1$, for $i = 0, \ldots, n-1$. For each of the 2^n labelings construct the eigenvalue by adding from the leaves to the root with the correct sign.

In order to construct the corresponding eigenvector, do exactly the same thing with the standard basis vector e_i in place of the coefficient a_i.

We illustrate the process for an 8 by 8 matrix with the sign labeling given by $s = (s_0, s_1, s_2) = (-1, 1, -1)$.

The corresponding eigenvector in this example is

$$e_0 - e_1 + e_2 - e_3 - (e_4 - e_5 + e_6 - e_7) = (1, -1, 1, -1, -1, 1, -1, 1).$$

The eigenvectors for a Walsh matrix are orthogonal since the matrices are symmetric. When constructed by the algorithm just described the eigenvectors contain entries of 1 and -1. Let us interpret each eigenvector as a piecewise constant function on the interval $[0, 1]$ where the ith component of the vector gives the value (1 or -1) of the function on the ith subinterval of length 2^{-n}. We have arrived at the orthogonal basis of **Walsh functions**, a countable orthogonal basis of the Hilbert space of square integrable functions (real or complex) on $[0, 1]$. See, for example, [1, 3, 4].

The algorithm described by the tree can be implemented by a short piece of code. This is True Basic, but it can be read as pseudo-code by users of other languages.

```
SUB eigenvalues_Walsh(a(),v())
 ! The input vector a() gives the coefficients of the matrix.
 ! The eigenvalues are written into the vector v().
 ! The length of a must be a power of 2 and indexed from 0.

    LET nn = ubound(a) - lbound(a) + 1  !length of a
    LET ss = 1
    DIM temp(0 to 1)                    !vector for temporary storage
    MAT temp = a                        !correct size of temp is set here
    DO
       FOR bb = 0 to nn-ss step 2*ss
           FOR j = 0 to ss-1 step 1
               LET v(bb + 2*j) = temp(bb + j) + temp(bb + j + ss)
               LET v(bb + 2*j + 1) = temp(bb + j) - temp(bb + j + ss)
           NEXT j
       NEXT bb
       LET ss = 2*ss
       MAT temp = v
    LOOP while ss < nn
END SUB
```

Remarks: For circulant matrices the algorithm to compute the eigenvalues from the coefficients is none other than the discrete Fourier transform. The algorithm here for Walsh matrices is the finite Walsh transform. Like the finite Fourier transform it can be seen as the product of a matrix and a vector, but it can be done with the number of steps on the order of $2^n n$ for matrices of size 2^n. (This is $m \log_2 m$ for $m \times m$ matrices.) In fact, the code presented is a "fast" version of the Walsh transform.

George Miminis [5] has considered another class of matrices with similar properties. Define the set $\mathcal{M}$ of square matrices by these two properties.

M1. $\mathcal{M}$ contains all 1 by 1 matrices.

M2. If $A, B \in \mathcal{M}$, then $\begin{bmatrix} A & -B \\ B & A \end{bmatrix} \in \mathcal{M}$.

It is possible to characterize matrices in $\mathcal{M}$ them in terms of dyadic addition with the complication of a plus or minus sign. An 8 by 8 example looks like this:

$$
\begin{bmatrix}
a_0 & -a_1 & -a_2 & a_3 & -a_4 & a_5 & a_6 & -a_7 \\
a_1 & a_0 & -a_3 & -a_2 & -a_5 & -a_4 & a_7 & a_6 \\
a_2 & -a_3 & a_0 & -a_1 & -a_6 & a_7 & -a_4 & a_5 \\
a_3 & a_2 & a_1 & a_0 & -a_7 & -a_6 & -a_5 & -a_4 \\
a_4 & -a_5 & -a_6 & a_7 & a_0 & -a_1 & -a_2 & a_3 \\
a_5 & a_4 & -a_7 & -a_6 & a_1 & a_0 & -a_3 & -a_2 \\
a_6 & -a_7 & a_4 & -a_5 & a_2 & -a_3 & a_0 & -a_1 \\
a_7 & a_6 & a_5 & a_4 & a_3 & a_2 & a_1 & a_0
\end{bmatrix} \tag{2}
$$

Theorem 3. *A square matrix of size 2^n is in $\mathcal{M}$ if and only if its i, j entry is $(-1)^{\sigma(i,j)} a_{i \oplus j}$, where $\sigma(i, j)$ is the number of locations in the binary representations of i and j in which i has a 0 and j has a 1.*

Proof. If we disregard signs, then we have a Walsh matrix. The sign of an entry is determined by how many times it falls into the upper right corner of the 2 by 2 block structures as we proceed from the full matrix down to 1 by 1 matrices. That happens each time the column index has a 1 and the row index has a 0 in the corresponding binary representation of the indices. □

As with Walsh matrices, the matrices of Miminis commute if they have the same size. So, the following theorem allows us to compute eigenvalues and eigenvectors.

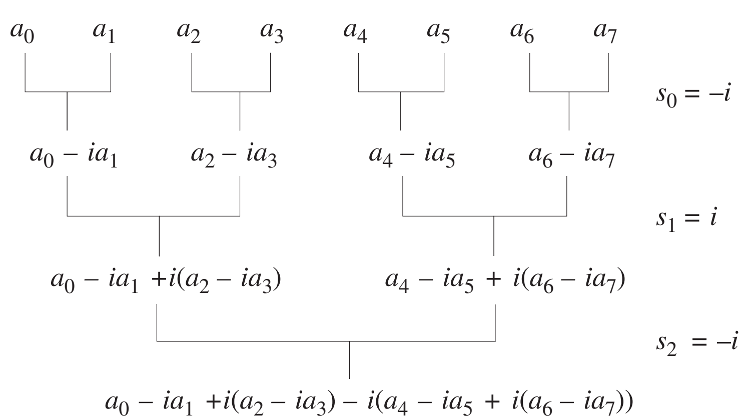
Theorem 4. *If A and B are commuting matrices and v is an eigenvector of both A and B with eigenvalues λ and μ, respectively, then (v, iv) and $(v, -iv)$ are eigenvectors of*

$$
\begin{bmatrix} A & -B \\ B & A \end{bmatrix}
$$

with eigenvalues $\lambda + i\mu$ and $\lambda - i\mu$, respectively.

Proof. Left to the reader. □

Thus, with minor modifications the algorithm used for Walsh matrices works for those of Miminis. Use $\pm i$ in place of ± 1 in the labeling of the signs. Let's rework the tree in Figure 1.



The eigenvalue simplifies to

$$a_0 - ia_1 + ia_2 + a_3 - ia_4 - a_5 + a_6 - ia_7.$$

The associated eigenvector is

$$(1, -i, i, 1, -i, -1, 1, -i).$$

The code for the eigenvalues of a Walsh matrix can be changed slightly to handle Miminis's matrices. Now we assume that the coefficients a_k are actually complex and we store them in a two dimensional array `a(k,j)`, $0 \le k \le 2^n - 1$, $j = 1, 2$, where `a(k,1)` is the real part of a_k and `a(k,2)` is the imaginary part.

```
SUB eigenvalues_Miminis(a(,),v(,))
 ! The input array a(,) gives the coefficients of the matrix.
 ! The eigenvalues are written into the vector v(,).
 ! The first index of a(,) must run from 0 to 2^n - 1.
 ! a(k,1) and a(k,2) are the real and imaginary parts.

    LET nn = (ubound(a) - lbound(a) + 1)/2  !number of coefficients
    LET ss = 1
    DIM temp(0 to 1,2)                    !array for temporary storage
    MAT temp = a                          !correct size of temp is set here
    DO
       FOR bb = 0 to nn-ss step 2*ss
           FOR j = 0 to ss-1 step 1
               LET v(bb + 2*j,1) = temp(bb + j,1) - temp(bb + j + ss,2)
               LET v(bb + 2*j,2) = temp(bb + j,2) + temp(bb + j + ss,1)
               LET v(bb + 2*j + 1,1) = temp(bb + j,1) + temp(bb + j + ss,2)
               LET v(bb + 2*j + 1,2) = temp(bb + j,2) - temp(bb + j + ss,1)
           NEXT j
       NEXT bb
       LET ss = 2*ss
       MAT temp = v
    LOOP while ss < nn
END SUB
```

References

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