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CONNECTED COMPONENTS OF REPRESENTATION VARIETIES

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ABSTRACT. This is primarily a survey of results known about the connected components of varieties of representations of groups and algebras. The examples discussed are finite groups, finite dimensional algebras, commutative algebras, finite dimensional Lie algebras, finitely generated nilpotent groups, fundamental groups of compact orientable surfaces.

Let Γ be a finitely generated discrete group or let A be a finitely generated associative k-algebra over a field k. The set of n-dimensional representations of Γ or A, that is, either $\mathrm{Hom}_{grp}(\Gamma,GL_n(k))$ or $\mathrm{Hom}_{k\text{-alg}}(A,M_n(k))$, is a variety defined over k. The basic problem of interest in this article is the characterization of the connected components of these varieties in terms of algebraic data coming from Γ or A, or, as we shall see with surface groups, in terms of topological data -- characteristic classes -- of flat bundles on the surface. Since these spaces of representations are algebraic varieties, for each n there are only a finite number of connected components.

Let $R_n(\Gamma)$ and $R_n(A)$ denote these algebraic varieties. (This is the notation of [LM] which will be used in this article.) They are affine varieties over k, which we take to be algebraically closed. We will generally use $R_n(A)$ in what follows. The reader should realize that setting $A = k\Gamma$, the group algebra of Γ, will give us $R_n(\Gamma)$. Two representations are isomorphic when they are in the same orbit of the conjugation action of $GL_n(k)$ on $R_n(A)$. Each orbit is connected since it is the image of the connected group $GL_n(k)$ under the orbit map, so we need only consider the connected components of the orbit space $R_n(A)/GL_n(k)$. This, however, is not usually a nice space but there is a better quotient; it is the variety associated to the ring of invariants $\mathcal{O}(R_n(A))^{GL_n(k)}$. Its points are in one-to-one correspondence with

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the isomorphism classes of semi-simple representations, these being precisely the closed orbits of the action, and is denoted $SS_n(A)$. See [LM, Pr]. When Γ is a fundamental group topologists as in [CS] are calling $SS_n(\Gamma)$ the "character variety" of n-dimensional representations, a suggestive name because the traces of the elements $\rho(\gamma)$, for $\gamma \in \Gamma$ fixed and ρ varying over representations, generate the coordinate ring of $SS_n(\Gamma)$. For this quotient the connected components are also in one-to-one correspondence with those of $R_n(\Gamma)$, the reason being that a component of $R_n(\Gamma)$ is determined by a minimal idempotent in its coordinate ring and idempotents are $GL_n(k)$-invariant since $GL_n(k)$ is connected. So we have reduced the main question to this: when are two semi-simple representations in the same component of $SS_n(A)$?

When $k = \mathbb{C}$ we can consider the usual (strong) topology on $R_n(A)$ and $SS_n(A)$ and we can ask about the strongly connected components and the path components. Happily all three notions -- the Zariski components, the strong components, the path components -- coincide for complex algebraic varieties. It is easy to see that path connected $\Rightarrow$ strongly connected $\Rightarrow$ Zariski connected. To show Zariski connected $\Rightarrow$ path connected, first note that a Zariski connected component is the union of irreducible components with a connected incidence graph. Vertices represent irreducible components and two vertices are connected by an edge if the corresponding irreducible components meet. Then use the fact that two points in an irreducible variety lie in an irreducible curve, which over $\mathbb{C}$ is a Riemann surface, possibly singular, but certainly path connected.

Now the problem of classifying the components should be easier than that of classifying all finite dimensional representations and even easier, as we have seen, than classifying the semi-simple representations. In the classical moduli problems of algebraic geometry there are two aspects: first to describe the discrete invariants (genus of a curve, degree and rank of a vector bundle, Hilbert polynomial) and then to describe the continuous invariants (construction of a good moduli space). The discrete invariants are usually easy to see and understand while the continuous invariants require all the work. But for general representation varieties we have a good moduli space $SS_n(A)$, although we do not know too much about it, but we do not even know the discrete invariants, i.e. the connected components.

In order to put some algebraic structure into the picture one may collect together the connected components of $R_n(A)$ for all n in $\mathbb{N}$ and make them into an abelian monoid using direct sum for addition. Then form the associated group by adjoining formal inverses and denote this group $D(A)$. In the case of representations of a group Γ the tensor product provides a multiplication to make $D(\Gamma)$ a commutative ring, and $D(\Gamma)$ is a quotient of the usual representation ring $R[\Gamma]$. In fact, by topologizing $R[\Gamma]$ with the natural topology obtained by constructing $R[\Gamma]$ as the group associated to the monoid $\bigcup_{n\geq 0} R_n(\Gamma)/GL_n(k)$ (disjoint union), we see that $D(\Gamma)$ is the ring of connected components, $R[\Gamma]$ modulo the connected component containing 0.

Intermediate between $R[\Gamma]$ and $D(\Gamma)$ is $K_0(\Gamma\text{-}\underline{\text{mod}})$ where Γ-$\underline{\text{mod}}$ is the category of finite dimensional representations of Γ. There are surjective homomorphisms $R[\Gamma] \twoheadrightarrow K_0(\Gamma\text{-}\underline{\text{mod}}) \twoheadrightarrow D(\Gamma)$. Now $K_0(\Gamma\text{-}\underline{\text{mod}})$ as a group is free abelian on the classes of simple modules and the class of ρ in $D(\Gamma)$ only depends on the image of ρ in $SS_n(\Gamma)$, which is the isomorphism class of the semi-simplification of ρ, that is, the sum of the simple modules in a composition series for ρ. Therefore $K_0(\Gamma\text{-}\underline{\text{mod}})$ maps surjectively onto $D(\Gamma)$. Likewise the corresponding statements hold for the groups $R[A]$, $K_0(A\text{-}\underline{\text{mod}})$ and $D(A)$.

A word of warning: it is not known in general that the cancellation property holds for the monoid of components, so it is not known whether we lose any information in the construction of the group $D(A)$. We do not know whether it is possible for $\rho \oplus \sigma$ to be in the same component as $\rho \oplus \tau$ while σ and τ are in different components. In all known examples, however, cancellation does in fact hold.

The basic data are collected in the following examples.

Example 1 Let A be a finite dimensional k-algebra. Each component of $R_n(A)$ contains a unique closed orbit of semi-simple representations, so that the connected components are bijective with the isomorphism classes of semi-simple representations. Two representations are in the same component if and only if they have isomorphic simple factors in their composition series. If A is semi-simple then the components are exactly the orbits and also the irreducible components; this is the situation for representations of a finite group Γ when the characteristic of k does not divide the order of Γ. We have $D(A) = R[A]$ and $D(\Gamma) = R[\Gamma]$; the quotient varieties $SS_n(A)$ and $SS_n(\Gamma)$ are finite sets of isolated points [Ar, Ga, Pr].

Example 2 A is a finitely generated commutative k-algebra, k algebraically closed. A splits into a product of connected algebras (i.e. having no non-trivial idempotents or equivalently a connected spectrum) $A = A_1 \times \ldots \times A_s$. An A-module decomposes into a sum of A_i-modules and $SS_n(A)$ is the disjoint union of $SS_{n_1}(A_1) \times \ldots \times SS_{n_s}(A_s)$ as $(n_1, \ldots, n_s) \in \mathbb{N}^s$ ranges over s-tuples with $n_1 + \ldots + n_s = n$. Then $SS_m(A_i) = (\text{Spec } A)^m/\Sigma_m$ because the simple A_i-modules are one-dimensional and $R_1(A_i) = SS_1(A_i) = \text{Spec } A_i$; Σ_m is the symmetric group acting on the m-fold product of Spec A by permuting the factors. Then $SS_m(A_i)$ is connected so that the components of $SS_n(A)$ are identified by $(n_1, \ldots, n_s)$ which is a dimension vector. Cancellation holds in the monoid and $D(A) = \mathbb{Z}^s$ [Mo 1].

Example 3 $A = U(\mathfrak{g})$, the enveloping algebra of a finite dimensional Lie algebra $\mathfrak{g}$ over a field k, algebraically closed of characteristic zero. In this case we write $R_n(\mathfrak{g})$, and so on, in place of $R_n(U(\mathfrak{g}))$. The components of $R_n(\mathfrak{g})$ are bijective with the components of $R_n(\mathfrak{g}/\text{rad } \mathfrak{g})$, via the natural map $R_n(\mathfrak{g}/\text{rad } \mathfrak{g}) \to R_n(\mathfrak{g})$, and these are precisely the isomorphism classes of $\mathfrak{g}/\text{rad } \mathfrak{g}$ - modules since $\mathfrak{g}/\text{rad } \mathfrak{g}$ is semi-simple. Again cancellation holds in the monoid of connected components since it is inherited from the category of $\mathfrak{g}/\text{rad } \mathfrak{g}$-modules. $D(\mathfrak{g})$ as a group is free abelian on the classes of simple $\mathfrak{g}/\text{rad } \mathfrak{g}$-modules. [Mo 1].

Examples 1-3 have an underlying similarity. In each case there is a subalgebra $B \subset A$ such that $R_n(A) \to R_n(B)$ induces a bijection on connected components and B is a semi-simple algebra so that the components of $R_n(B)$ are just the orbits. Thus $SS_n(B)$ is a finite set of points and the inverse images of the map $SS_n(A) \to SS_n(B)$ are the connected components. In example 1, B is a semi-simple algebra isomorphic to A/N where N is the nilpotent radical. In example 2, $B = k \times \ldots \times k$, s factors, embedded in

$A = A_1 \times \ldots \times A_s$. In example 3, $B = U(\mathfrak{s})$ where $\mathfrak{s}$ is the semi-simple factor in the Levi decomposition $\mathfrak{g} = \mathfrak{s} \oplus \mathrm{rad}\,\mathfrak{g}$ [Mo 1].

Example 4 Γ is a nilpotent group and Γ^{ab} is torsion free (Γ^{ab} is the abelianization $\Gamma/[\Gamma,\Gamma]$.) The components of $SS_n(\Gamma)$ are the twist-isomorphism classes and they are the irreducible components, too. Let $\rho = \rho_1 \oplus \ldots \oplus \rho_m$ be a semi-simple representation decomposed into simple ones. Let $\chi = (\chi_1, \ldots, \chi_m)$ be a multi-character; i.e. each χ_i is a one-dimensional representation. Then $\chi \bullet \rho = \chi_1\rho_1 \oplus \ldots \oplus \chi_m\rho_m$ is the twist of ρ by χ and is said to be "twist-isomorphic" to ρ. The twist of a simple representation remains simple so $D(\Gamma)$ is generated by the twist-isomorphism classes of simple representations, and in fact freely generated by them, the argument being that the monoid of connected components is the free abelian monoid on the twist-isomorphism classes of simple representations. If $\rho = \rho_1 \oplus \ldots \oplus \rho_m$ and $\sigma = \sigma_1 \oplus \ldots \oplus \sigma_l$ are in the same component then each simple factor ρ_i is twist isomorphic to some simple σ_j and $l = m$. The monoid is $\mathbb{N}^{(S)}$ with S the set of twist-isomorphism classes of simple representations and $D(\Gamma) = \mathbb{Z}^{(S)}$ [LM, Ru].

Whereas the components in these four examples are given by algebraic data in the rest of the examples topological data -- characteristic classes or classes of vector bundles -- prove effective. We also consider representations into groups other than $GL_n(k)$ such as $PSL_2(\mathbb{R})$ and $U(n)$.

Let M be a manifold. For each homomorphism ρ from the fundamental group of M to a Lie group G there is a principal G-bundle P over M with a flat connection and also flat connections on any associated bundles. Thus $GL_n(\mathbb{C})$ representations give flat complex vector bundles. The construction of the principal bundle is this: $\pi_1(M)$ acts on $\tilde{M} \times G$ by $\gamma \bullet (z,g) = (z\gamma^{-1}, \rho(\gamma)g)$, where $\tilde{M}$ is viewed as a principal $\pi_1(M)$-bundle, and $P = (\tilde{M} \times G)/\pi_1(M)$. The flat connection on P, viewing it as a horizontal distribution, descends from the trivial flat connection on $\tilde{M} \times G$ whose leaves are $\tilde{M} \times \{g\}$, and ρ is the holonomy or the monodromy of the connection. The term holonomy is preferred by differential geometers and monodromy is preferred by algebraic geometers. If ρ is changed to an isomorphic representation the resulting flat connection is isomorphic, using the appropriate notion of morphism of connections, to the flat connection arising from ρ, and every flat connection on a principal G-bundle can be constructed from its holonomy (monodromy). Therefore there is a bijection between $\mathrm{Hom}(\pi_1(M),G)/G$, the isomorphism classes of representations, and the isomorphism classes of flat connections on all principal G-bundles over M.

Key point If ρ_0 and ρ_1 are in the same component of $\mathrm{Hom}(\pi,G)$ then there is a homotopy of the associated principal bundles so that the bundles must be isomorphic (disregarding the flat connections). Thus any topological invariant of G-bundles is a discrete invariant of representations into G.

First and foremost among these invariants is simply the topological type of the bundle and is the main theme of the remaining examples. The topological type may be expressed as an Euler class or a Stiefel-Whitney class or a Chern class, particularly when the base space is a surface, in which case it can also be seen as the obstruction to lifting the representation $\pi_1 \to G$ to the universal cover $\tilde{G}$.

When G is an algebraic group (real or complex), as in all our examples, $\mathrm{Hom}(\pi,(M),G)$ has a finite number of components, since it is an algebraic variety, so there are only a finite number of distinct principal G-bundles that have flat connections. Understanding which G-bundles have flat connections is part of the problem of classifying the connected components of $\mathrm{Hom}(\pi,G)$ using topological information. The components (finite in number) are grouped into classes with isomorphic underlying G-bundles so that the classification problem has two parts:

(1) Identify the G-bundles admitting flat connections;

(2) For each such G-bundle indentify the components associated to it.

Example 5 We will see this program explicitly here for M a compact orientable surface of genus $g \geq 2$ and $G = PSL_2(\mathbb{R})$ or $SL_2(\mathbb{R})$. Goldman [Go 1] and Jankins [Ja] determined that the connected components of $\mathrm{Hom}(\pi_1(M), PSL_2(\mathbb{R}))$ are exactly the fibers of the continuous map $e : \mathrm{Hom}(\pi_1(M), PSL_2(\mathbb{R})) \to \mathbb{Z}$, the "Euler number" map defined so that $e(\rho)$ is the Euler number of the $\mathbb{R}\mathbb{P}^1$-bundle associated to the principal $PSL_2(\mathbb{R})$-bundle arising from ρ where $PSL_2(\mathbb{R})$ acts on $\mathbb{R}\mathbb{P}^1 = S^1$ in the natural way as linear fractional transformations on $\mathbb{R} \cup \{\infty\}$, or equivalently as automorphisms of $\mathbb{R}\mathbb{P}^1$ as the set of lines in $\mathbb{R}^2$. It was already known by work of Wood [Wo] that an S^1-bundle with structure group $\mathrm{Diff}^+ S^1$ has a flat connection if and only if its Euler number e satisfied $|e| \leq |\chi(M)| = 2g - 2$. (Beware that these are not principal S^1-bundles under consideration.) Earlier Milnor had shown that a principal bundle with structure group $GL_2^+(\mathbb{R})$, the plus denoting positive determinant, has a flat connection if and only if its Euler number (that of the associated plane bundle) e satisfies $|e| \leq g - 1$ [Mi]. Now for $GL_2^+(\mathbb{R})$ we may just as well use $SL_2(\mathbb{R})$. The factor of 2 present in the first inequality can be explained by the fact that $q : SL_2(\mathbb{R}) \to PSL_2(\mathbb{R})$ is a double cover.

We have the commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}(\pi_1(M), SL_2(\mathbb{R})) & \xrightarrow{q_*} & \mathrm{Hom}(\pi_1(M), PSL_2(\mathbb{R})) \\ e' \downarrow & & \downarrow e \\ \mathbb{Z} & \xhookrightarrow{2} & \mathbb{Z} \end{array}$$

where e and e' are the Euler class maps and 2 denotes multiplication by 2. Goldman [Go 2] has also determined the components of $\mathrm{Hom}(\pi_1(M), SL_2(\mathbb{R}))$. For $1 - g < k < g - 1$, $(e')^{-1}(k)$ is a component and is a covering space of $e^{-1}(2k)$ of index 2^{2g}. For $k = 1 - g$ and $k = g - 1$, $(e')^{-1}(k)$ is 2^{2g} components, each one of which is diffeomorphic to $e^{-1}(2k)$. The difference for the extreme value of k is due to the fact that the components of $\mathrm{Hom}(\pi_1(M), PSL_2(\mathbb{R}))$ with Euler number $\pm(2 - 2g)$ correspond to Fuchsian representations of $\pi_1(M)$ and their orbit spaces correspond to Teichmüller space which is contractible. Thus we see that there are $2^{2g+1} + 2g - 3$ components of $\mathrm{Hom}(\pi_1(M), SL_2(\mathbb{R}))$ partitioned into $2g - 1$ topological types.

More can be said about the topology of the other components of $\mathrm{Hom}(\pi_1(M), PSL_2(\mathbb{R}))$. Hitchin [Hi] has recently shown that the moduli space $e^{-1}(k)/PSL_2(\mathbb{R})$ is diffeomorphic to a complex vector bundle of $\mathbb{C}$-rank $g - 1 + k$ over the symmetric product $S^{2g-2-k}M$.

The Euler number can be seen more directly for a representation $\rho : \pi_1(M) \to G$ as an obstruction to lifting ρ to $\tilde{\rho} : \pi_1(M) \to \tilde{G}$, where $\tilde{G}$ is the universal covering group of G. Let $\pi_1(M)$ have the standard presentation $\langle \alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g : \Pi [\alpha_i, \beta_i] = 1 \rangle$. Let $\rho(\alpha_i) = A_i$, $\rho(\beta_i) = B_i$ in G. Pick any lifts of A_i and B_i in $\tilde{G}$ and call them $\tilde{A}_i$ and $\tilde{B}_i$. Then $\Pi [\tilde{A}_i, \tilde{B}_i] \in \mathrm{Ker}\, p = \pi_1(G)$, $p : \tilde{G} \to G$. As $\pi_1(PSL_2(\mathbb{R})) = \mathbb{Z}$ we get an integer which is the Euler number. For an arbitrary manifold M we consider the exact sequence of Čech cohomology sets

$$H^1(M,\tilde{G}) \to H^1(M,G) \to H^2(M,\pi_1(G))$$

arising from the exact sequence of constant sheaves over M

$$0 \to \pi_1(G) \to \tilde{G} \to G \to 1,$$

and we interpret $H^1(M,G)$ as the isomorphism classes of principal G-bundles with flat connections, i.e. $H^1(M,G) = \mathrm{Hom}(\pi_1(M),G)/G$. The obstruction class map o_2 is the composition

$$\mathrm{Hom}(\pi_1(M),G) \to \mathrm{Hom}(\pi_1(M),G)/G \xrightarrow{\approx} H^1(M,G) \to H^2(M,\pi_1(G)).$$

For $G = U(n)$ or $GL_n(\mathbb{C})$ we get the first Chern class, an integral cohomology class since $\pi_1(U(n)) = \mathbb{Z}$. For $G = SO(n)$, $n \geq 3$, we get the second Stiefel-Whitney class in $H^2(M, \mathbb{Z}_2)$, whose vanishing means the structure group lifts to $\mathrm{Spin}(n)$.

In 1976 Heitsch posed the problem of finding the supremum of the absolute value of the Euler number for real vector bundles of rank $2n$ with structure group a discrete subgroup of $SL_{2n}(\mathbb{R})$ i.e. flat orientable bundles, over a manifold M of dimension $2n$ and to do so in terms of topological invariants of M such as the Euler and Pontryagin classes [DT]. Earlier Sullivan showed that an upper bound is the number of top dimensional simplices of a triangulation of M [Su]. See also [Du, He].

Example 6 M is a compact surface of genus g just as in the previous example. The space of equivalence classes of unitary representations $\mathrm{Hom}(\pi_1(M),U(n))/U(n)$ is the moduli space of semi-stable holomorphic vector bundles of degree 0 (i.e. topologically trivial) and rank n on a Riemann surface with M as the underlying differentiable manifold [NS, Se]. A priori $\mathrm{Hom}(\pi_1(M),U(n))/U(n)$ has no complex structure but in fact it is a projective variety and the complex structure depends on the complex structure of M, an aspect described in greater generality in [AB, H]. When $n = 1$, the $U(1)$ conjugation action is trivial, and $\mathrm{Hom}(\pi_1(M),U(1))$ is the Jacobian variety of the Riemann surface. Topologically, $\mathrm{Hom}(\pi_1(M),U(1)) = \mathrm{Hom}(H_1(M,\mathbb{Z}),U(1)) = \mathrm{Hom}(\mathbb{Z}^{2g},U(1)) = U(1)^{2g}$, but of course this ignores the complex structure [Gu 1, Gu 2].

$\mathrm{Hom}(\pi_1(M),U(n))$ is connected as shown by the work of Narasimhan and Seshadri. The quotient space fibers over the Jacobian variety of M; the projection is the map that sends each unitary representation to its determinant. The fibers are irreducible varieties. It should be noted that since $U(n)$ is compact each representation is semi-simple and so each orbit is closed and $\mathrm{Hom}(\pi_1(M),U(n))/U(n)$ is automatically Hausdorff.

Example 7 For bundles of non-zero degree Atiyah and Bott [AB] have shown that there is a central extension of $\pi_1(M)$ by $\mathbb{R}$ that they denote $\Gamma_{\mathbb{R}}$ which generalizes the role of the fundamental group in degree zero. The orbit space $\mathrm{Hom}(\Gamma_{\mathbb{R}},U(n))/U(n)$ is the moduli space of all semi-stable holomorphic vector bundles of rank n over M. The degree (first Chern class) is a discrete invariant and there is exactly one connected component for each degree. Thus there are an infinite number of components, but $\mathrm{Hom}(\Gamma_{\mathbb{R}},U(n))$ is not an algebraic variety since $\Gamma_{\mathbb{R}}$ is not finitely generated. It is defined by the exact sequence $0 \to \mathbb{R} \to \Gamma_{\mathbb{R}} \to \pi_1(M) \to 1$ with $\prod [\alpha_i,\beta_i] = 1 \in \mathbb{R}$. For $n = 1$ every homomorphism $\Gamma_{\mathbb{R}} \to U(1)$ factors through $\Gamma_{\mathbb{R}}^{ab} \cong U(1) \times \mathbb{Z}^{2g}$ so $\mathrm{Hom}(\Gamma_{\mathbb{R}},U(1)) = \mathrm{Hom}(U(1) \times \mathbb{Z}^{2g},U(1)) \cong \mathbb{Z} \times U(1)^{2g}$ with the $\mathbb{Z}$-factor specifying the degree. This space, which is an abelian group, is the Picard group of M; the connected component containing 0 is the Jacobian variety.

Example 8 We know that $\mathrm{Hom}(\pi_1(M), GL_2(\mathbb{C})) = R_2(\pi_1(M))$ is connected for M a compact orientable surface of genus $g \geq 0$, and for $g = 0$ and 1, $R_n(\pi_1(M))$ is connected, but it is still unknown for $g \geq 2$ and $n \geq 3$ whether or not $R_n(\pi_1(M))$ is connected. The question can be reduced to the special linear representations:

If $\mathrm{Hom}(\pi_1(M), SL_n(\mathbb{C}))$ is connected for M a compact orientable surface then $\mathrm{Hom}(\pi_1(M), GL_n(\mathbb{C}))$ is connected. That is because the tensor product (twist) map $t : \mathrm{Hom}(\pi_1(M), \mathbb{C}^x) \times \mathrm{Hom}(\pi_1(M), SL_n(\mathbb{C})) \to \mathrm{Hom}(\pi_1(M), GL_n(\mathbb{C}))$ is surjective, since the equation $\chi \otimes \sigma = \rho$ can be solved for χ and σ, when ρ is given, by $\chi = (\det \rho)^{1/n}$ and $\sigma = \chi^{-1} \otimes \rho$. The nth roots of $\det \rho$ exist because $\det \rho$ factors through the free abelian group $H_1(M, \mathbb{Z})$ and maps to the divisible group $\mathbb{C}^x$. The proposition follows since $\mathrm{Hom}(\pi_1(M), \mathbb{C}^x) \cong (\mathbb{C}^x)^{2g}$ is connected, so the domain of t is connected.

In [Go 1, Go 2] it is shown that $\mathrm{Hom}(\pi_1(M), SL_2(\mathbb{C}))$ is connected by rather explicit computations and arguments that will not work for $SL_n(\mathbb{C})$ in general. However the proof of connectedness would follow easily from this stronger conjecture: the fibers of the commutator map

$$SL_n(\mathbb{C}) \times SL_n(\mathbb{C}) \to SL_n(\mathbb{C})$$
$$(A,B) \to ABA^{-1}B^{-1}$$

are connected. Then it would be possible to deform a representation σ considered as a 2g-tuple $(A_1, A_2, \ldots, A_g, B_1, \ldots, B_g)$ to the trivial representation by arbitrarily moving $A_1, \ldots, A_{g-1}, B_1, \ldots, B_{g-1}$ each to I and then deforming A_g and B_g so as to satisfy the defining relation. For $SL_2(\mathbb{C})$ it is true that these commutator fibers are connected, but the proof in [AM] does not generalize. There is only one isomorphism class of complex vector bundle arising from representations of $\pi_1(M)$ in $GL_n(\mathbb{C})$, namely the topologically trivial bundle, as contrasted with real vector bundles arising from $GL_2(\mathbb{R})$ representations. One explanation is that the Chern classes of a complex vector bundle with a flat connection are all torsion classes because as elements of $H^*(M, \mathbb{R})$ they can be constructed from the curvature form of a connection. Over an orientable surface there is no torsion and the Chern classes completely describe the bundles. The connectedness conjecture is equivalent to the claim that there is only one connected component of gauge-equivalence classes of flat connections on the trivial bundle $M \times \mathbb{C}^n$ when M is a surface.

The stable version of the conjecture is that $D(\pi_1(M)) = \mathbb{Z}$, and although weaker it is also unknown.

__Example 9__ This is not an example of known connected components but rather a general setting for the stable version of the connectedness conjecture for surfaces. For a finite group Γ there is a beautiful theorem of Atiyah [At 1] and for Γ a compact Lie group a generalization of Atiyah and Segal [AS] that relates the representation ring $R[\Gamma]$ with the ring of vector bundles $K(B\Gamma)$ over the classifying space of Γ. In particular the completion $\widehat{R[\Gamma]}$ with respect to the augmentaton ideal defined by the dimension map is isomorphic to $K(B\Gamma)$, which is complete by its construction as a limit of $K(B\Gamma_n)$ where $B\Gamma_n$ is the n-skeleton. When Γ is finite then $D(\Gamma)$ is the same as $R[\Gamma]$, but when Γ is discrete but not finite then the natural map $\alpha : R[\Gamma] \to K(B\Gamma)$ factors through $D(\Gamma)$, because the topological type of the bundle is constant on connected components. We denote this map by $\alpha : D(\Gamma) \to K(B\Gamma)$ which is also continuous for the augmentation ideal topology on $D(\Gamma)$, and thus there is a map $\hat{\alpha} : \widehat{D(\Gamma)} \to K(B\Gamma)$.

Consider the case that $\Gamma = \pi_1(M)$ for a surface M. Then $B\Gamma = M$ (which is also $K(\Gamma,1)$ since Γ is discrete). Now $K(M) = \mathbb{Z} \oplus \mathbb{Z}$ as a group and $\mathbb{Z}[T]/(T-1)^2$ as a ring where the bundle of rank n and degree (Chern class) d is $n + d(T-1)$. The image of α consists of the subring of flat bundles, i.e. those with $d = 0$, and is isomorphic to $\mathbb{Z}$. Since $D(\Gamma)$ contains a copy of $\mathbb{Z}$, knowing that α is injective would imply that $D(\Gamma) = \mathbb{Z}$. Merely knowing $\hat{\alpha}$ is injective would not give us that much without knowing the kernel of the completion $D(\Gamma) \to \widehat{D(\Gamma)}$.

However for a discrete group Γ the question is: to what extent is $\hat{\alpha} : \widehat{D(\Gamma)} \to K(B\Gamma)$ injective?

Another viewpoint for surface group representations in $GL_n(\mathbb{C})$ is through Chern classes. For a discrete group Γ and $\rho \in R_n(\Gamma)$ we can associate Chern classes $c_i(\rho) \in H^{2i}(\Gamma,\mathbb{Z}) = H^{2i}(B\Gamma,\mathbb{Z})$ which are simply the Chern classes of the associated vector bundle on $B\Gamma$ [At 1, Gr]. Obviously $c_i(\rho)$ only depends on the component containing ρ, and so $c_i(\rho)$ is a discrete invariant of $R_n(\Gamma)$. To what extent the Chern classes characterize the components is quite unknown, since even for Γ the fundamental group of a surface we do not know. (Then of course $c_1(\rho) = 0$ since it's a torsion class and $c_i(\rho) = 0$ for $i \geq 2$ since $B\Gamma = M$ has dimension 2.)

Example 10 This is another direction to generalize the conjecture that $\mathrm{Hom}(\pi_1(M),SL_n(\mathbb{C}))$ is connected for a surface M, replacing $SL_n(\mathbb{C})$ by an algebraic group G whose fundamental group is finite. As we have seen the obstruction to lifting $\rho : \pi_1(M) \to G$ to $\tilde{\rho} : \pi_1(M) \to \tilde{G}$ is given by $o_2 : \mathrm{Hom}(\pi_1(M),G) \to H^2(M,\pi_1(G)) = \pi_1(G)$. We may ask, as in [Go 2], to what extent are the fibers of o_2 the connected components? We have here two finite sets: the components of $\mathrm{Hom}(\pi_1(M),G)$ and $\pi_1(G)$ and also a map between them. The positive evidence that o_2 characterizes the components given in [Go 2] is for $G = SL_2(\mathbb{C})$ with $\pi_1(G) = 1$, as well as for $G = PSL_2(\mathbb{C})$ with $\pi_1(G) = \mathbb{Z}_2$, for $G = SU(2)$, $\pi_1(G) = 1$ and for $G = SO(3)$, $\pi_1(G) = \mathbb{Z}_2$. To this list we may add $G = SU(n)$, $\pi_1(G) = 1$, for all n, since the moduli space of special unitary representations $\mathrm{Hom}(\pi_1(M),SU(n))/SU(n)$ is connected by [Se]. It is the moduli space of semi-stable bundles of degree zero with holomorphically trivial determinant line bundle.

Example 11 For $SL_2(\mathbb{R})$ representations of a surface group we have seen that several connected components all give rise to the plane vector bundle with Euler number g-1 and 1-g. Examples of this phenomenon for $GL_n(\mathbb{C})$ representations and complex vector bundles are given by spherical space forms - manifolds of the form S^k/Γ where Γ is a finite group. Thus they are manifolds M whose universal cover is a sphere and with finite fundamental group and they include the real projective spaces and lens spaces. Except for the antipodal actions of $\mathbb{Z}_2$ on even dimensional spheres giving the even dimensional real projective spaces, the spherical space forms all come from a finite subgroup Γ of $U(k)$ acting on $S^{2k-1} \subset \mathbb{C}^{2k}$ without fixed points. The classification of spherical space forms may be found in [Wf]. What we need is their K-theory found in [At 2, Gi]. The homomorphism from the representation ring of Γ to $K(M)$ is surjective and the kernel is the ideal generated by $\sum_{i=0}^{k} (-1)^i [\wedge^i \tau]$ where τ is the natural representation of Γ on $\mathbb{C}^k$ (remember Γ is a subgroup of $U(k)$) and $\wedge^i \tau$ is the exterior power.

Now Γ is finite so the components of $R_n(\Gamma)$ are just the isomorphism classes of representations. If two non-isomorphic representations ρ and σ differ by $\Sigma(-1)^i[\wedge^i\tau]$ and if the dimensions are large enough so that the associated vector bundles are in the stable range, then the bundles arising from ρ and σ are in different components.

For a concrete example consider $M = \mathbb{R}\mathbb{P}^3 = S^3/\mathbb{Z}_2$. Then $R[\mathbb{Z}_2] = \mathbb{Z}[x]/(x^2-1)$ where x is the class of the non-trivial character. The representation $\tau : \mathbb{Z}_2 \to GL_2(\mathbb{C})$ maps the generator to $-I$. Thus $[\tau] = 2x$ in $R[\mathbb{Z}_2]$, and $[\wedge^0\tau] = 1$, $[\wedge^1\tau] = [\tau] = 2x$, $[\wedge^2\tau] = 1$. So $K(\mathbb{R}\mathbb{P}^3) \cong \mathbb{Z}[x]/(x^2-1, 2x-2)$. The rank 2 complex vector bundles arising from the trivial two-dimensional representation and from τ are the same in $K(\mathbb{R}\mathbb{P}^3)$, but that is in the stable range so they are isomorphic vector bundles and in fact the trivial bundle. The only two dimensional representations are the elements $2, 2x, x+1$ in $R[\mathbb{Z}_2]$, but $x+1$ defines a non-trivial bundle since $x+1 \neq 2$ in $K(\mathbb{R}\mathbb{P}^3)$, so we see that the rank 2 trivial bundle has exactly two components of gauge-equivalence classes of flat connections. while the other rank 2 bundle has one component. This phenomenon cannot occur for line bundles; $H^2(\mathbb{R}\mathbb{P}^3,\mathbb{Z}) = \mathbb{Z}_2$ classifies line bundles by the first Chern class and there are just two one-dimensional representations which give rise to distinct elements of $K(\mathbb{R}\mathbb{P}^3)$ and must each, therefore, correspond to one of the line bundles.

In general for any manifold M and any line bundle the space of all flat connections is connected. (Of course it may be empty.) This is because the space of flat connections of a line bundle is an affine space modeled on the space of closed 1-forms which is a connected space. We can also see that there is a one-to-one correspondence between the components of $\mathrm{Hom}(\pi_1(M),\mathbb{C}^x)$ and the torsion subgroup of $H^2(M,\mathbb{Z})$.

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