Discrete Yang-Mills Connections *

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Abstract

The computation of the Yang-Mills action is described for the discretization of the Hopf bundle $S^3 \to S^2$ proposed by Manton. The minimal discrete connections are described. By enlarging the structure group and lattice there is a unique minimal connection that is clearly a discrete analogue of a connection with a harmonic curvature form, which is a minimal connection in the continuous setting.

In a recent paper Manton [1] has described connections on discrete fiber bundles that retain vestiges of the topology of the continuous bundles being approximated. The two Hopf bundles $S^3 \to S^2$ and $S^7 \to S^4$ are discretized by finite lattices and the structure groups are replaced by finite subgroups of the structure groups $U(1)$ and $SU(2)$. A connection or gauge field can then be defined in a natural and geometric way was an equivariant lifting of the links in the base lattice. A brief summary of Manton's discretization of $S^3 \to S^2$ will help to make this clear. Consider S^2 as the Riemann sphere $\mathbf{C} \cup \{\infty\}$ with lattice points $0, \infty, 1, -1, i, -i$ and links joining all pairs except for 0 and , 1 and 1, i and $-i$. The resulting finite graph is the octahedron. Now consider the three-sphere as the unit sphere in $\mathbf{C}^2$ and the lattice of 24 points

$$(\pm 1, 0), (\pm i, 0), (0, \pm 1), (0, \pm i), \frac{1}{2}(\pm 1 \pm i, \pm 1 \pm i).$$

The projections maps (z, w) to z/w. Links are defined between the nearest neighbors measured in the standard norm of $\mathbf{C}^2$. Each point has eight nearest neighbors, two on each of the four fibers over the neighboring base points. Each fiber is the orbit of the group $\{1, -1, i, -i\} \cong \mathbf{Z}_4$ of fourth roots of unity.

A connection is an equivariant lifting of the base links. There are two ways to lift each base link because each point in the total lattice is linked to two points in the fiber at the other end of the base link. The equivariance means that we only need to keep track of what happens to one point in each fiber. Since there are 12 edges, there are $2^{12} = 4096$ different connections and all are gauge inequivalent because the structure group $\mathbf{Z}_4$ is abelian. The

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holonomy around any closed path of links in the base is an element of $\mathbf{Z}_4$. Associated to each of the oriented faces of the octahedron is the holonomy around its boundary. Define the phases of $1, -1, i, -i$ to be $0, \pi, \pi/2, \pi/2$, respectively.

The phase of the holonomy around a face is the discrete analogue of the curvature of the connection, i.e. the field strength. Manton shows that there is no ambiguity in this example in using π for the phase of 1 and that the sum over the eight faces of the holonomy phases, which may appropriately be called the total flux, is 2π for any of the connections. Therefore it can be said that the Chern number of this discrete bundle is 1. Other approaches to lattice gauge theory require the avoidance of certain exceptional configurations in order to have a well-defined topological type. See [2, 3, 4].

Now to push what Manton has done a bit further we consider the discrete version of the Yang-Mills action for the this discrete bundle. Define the action to be the sum of the squared curvature (i.e. flux) over the faces. With a computer program written in Mathematica and running on a NeXT work-station the action has been computed for all 2^{12} connections. The minimum value is π^2 and there are 450 minimal connections. For each of the minimal connections the flux is 0 on four of the faces and $\pi/2$ for the remaining four faces. The following table shows the values and the number of connections with each value.

Yang-Mills Values

value(units of π^2)	number of connections
1.0	450
1.5	960
2.0	1432
2.5	768
3.0	396
3.5	64
4.0	24
4.5	0
5.0	2

How well does a minimal discrete connection approximate a minimal smooth connection? A $U(1)$-connection on a compact Riemannian manifold is minimal if and only if its curvature is a harmonic 2-form, using the identification of the Lie algebra of $U(1)$ with the real numbers. On a surface the curvature is a constant multiple of the volume 2-form. If S^2 has the standard round metric and is divided into eight congruent spherical triangles, then a harmonic 2-form has the same integral over each of the triangles, and so on the lattice version of S^2 a minimal connection ought to have the flux value of $\pi/4$ on each of the eight faces.

This can in fact be achieved by a natural enlargement of the lattice in S^3. Just double the size of the lattice by using the eighth roots of unity $\{e^{2\pi ik/8} | k = 0, \ldots, 7\}$ for the structure group, so that each fiber is the $\mathbf{Z}_8$-orbit in $\mathbf{C}^2$ of any of the original points. Then add new links between the new nearest neighbors, while retaining the old links. Each point in the S^3 lattice has a unique nearest neighbor in each of the neighboring fibers at a distance of

$\sqrt{2-\sqrt{2}} \approx 0.765$. For example, $\frac{1}{2}(1+i, 1+i)$ is in the fiber over 1 and its nearest neighbor in the fiber over ∞ is $(e^{\pi i/4}, 0)$, one of the new points. In the enlarged lattice there are three ways to lift each base link, and so there are $3^{12} = 531,441$ discrete connections. The total flux of any of the 3^{12} connections is still 2π, because any new connection can be constructed from one of the old connections by changing the lifting of one link at a time. Whenever an old link is replaced by one of the new ones the holonomy around one face changes by $e^{\pi i/4}$ and around another face it changes by $e^{-\pi i/4}$. Thus the total flux remains the same. Now, however, there is a unique minimal connection, namely the one defined by lifting base links to the links connecting the nearest neighbors. Thus, if v and w are linked in the base lattice and p is in the fiber over v, then there is a unique point q at a distance of 0.765 from p in the fiber over w. A computer calculation shows that the holonomy around any of the faces is $e^{\pi i/4}$, so that the flux through any face is $\pi/4$. Therefore, this minimal connection is just what a discrete harmonic connection should be.

References

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