

GEOMETRIC EQUIVALENCE RELATIONS ON MODULES

Kent MORRISON

*Department of Mathematics, California Polytechnic State University, San Luis Obispo, CA
93407, USA*

Communicated by H. Bass
Received June 1980

0. Introduction

In [4] Gersten developed the notion of homotopy for ring homomorphisms. A simple homotopy of two ring homomorphisms $f, g: A \rightarrow B$ can be viewed as a deformation over the parameter space $\text{Spec } \mathbf{Z}[t]$. This is a homomorphism $A \rightarrow B[t]$ which restricts to f when $t=0$ and to g when $t=1$. Two homomorphisms f and g are homotopic if there is a chain of homomorphisms starting with f and ending with g such that each term is simple homotopic to the next.

Let A be a k -algebra and k a field. Consider the finite dimensional representations of A and require that a simple homotopy of representations be given by a deformation over $\text{Spec } k[t] = \mathbf{A}_k^1$. Using direct sum we can make the homotopy classes of representations into an abelian monoid. Now it is more useful to use any nonsingular, rational affine curve as well as \mathbf{A}_k^1 for the parameter space of a homotopy. (This becomes apparent when A is commutative; see Section 1.3.) The abelian monoid of homotopy classes is denoted by $H(A)$ and its associated group by $R(A)$. Two modules (representations) whose classes in $R(A)$ are the same are said to be 'rationally equivalent'. In Section 1.3 we show that when A is commutative $R(A)$ is isomorphic to the Chow group of 0-cycles of $\text{Spec } A$ modulo rational equivalence. In Section 1.2 we prove basic structure theorems about the functor H from k -algebras (finitely generated) to abelian monoids.

Now consider more general deformations using any connected affine k -scheme as the parameter space. The result is a coarser equivalence relation on the finite dimensional A -modules and a corresponding abelian monoid which we denote $C(A)$. In Section 1.1 we determine $C(A)$ for some algebras: finite dimensional algebras, enveloping algebras, commutative algebras. We prove basic structure theorems about $C(A *_k B)$, $C(A \otimes_k B)$, $C(A \times B)$. Let $D(A)$ be the associated abelian group. Two finite dimensional modules M and N are said to be 'algebraically

equivalent' if their classes in $D(A)$ are the same. In Section 1.3 we show that when A is commutative and k algebraically closed, $D(A)$ is isomorphic to the group of 0-cycles of $\text{Spec } A$ modulo algebraic equivalence.

The results on algebraic and rational equivalence in Section 1.3 suggest that $D(A)$ and $R(A)$ are good noncommutative generalizations of the groups of 0-cycles modulo algebraic and rational equivalence.

In Section 2 we consider the category of finitely generated (left) A -modules and the equivalence relations obtained by deformations over rational curves and over arbitrary connected k -schemes. The main result here is that the group of rational equivalence classes is isomorphic to $K'_0(A)$ when A is left-noetherian.

1. Finite dimensional modules

1.1. The monoid $C(A)$ and the group $D(A)$

Let A be a finitely generated algebra over the field k . The functor $\text{Mod}_A(m)$ is representable, where $\text{Mod}_A(m)(R)$ is the set of left $A \otimes R$ -module structures on R^m and R is a commutative k -algebra. See [3] or [5]. Let Mod_A denote the union over $m \in \mathbb{N}$ so Mod_A is locally algebraic (locally of finite type) over k . The direct sum of modules gives rise to a scheme morphism

$$\oplus : \text{Mod}_A \times \text{Mod}_A \rightarrow \text{Mod}_A$$

which is additive on the dimensions. We let $C(A)$ denote the monoid whose elements are the connected components of Mod_A and whose addition is inherited from \oplus . We may speak of two modules in the same component as C -equivalent and we denote by $\langle M \rangle$ the connected component of M as an element of $C(A)$.

The dimension function $\dim : C(A) \rightarrow \mathbb{N}$ is a homomorphism of monoids and provides a natural augmentation for $C(A)$. Let $C_m(A)$ be the classes of modules of dimension m .

We make $C(A)$ into a group in the standard way by adjoining additive inverses. Let $D(A)$ denote the group. The natural map from $C(A)$ to $D(A)$ may not be injective so the equivalence relation on modules arising from $D(A)$ may be coarser than C -equivalence. Two modules M and N are in the same class in $D(A)$ if there exists a third module P such that $M \oplus P$ and $N \oplus P$ are in the same connected component of $C(A)$. This is the coarsest geometric equivalence relation that we examine in this paper.

Since $D(A)$ provides less information we concentrate on $C(A)$ for the rest of this section.

C is a contravariant functor from the category of finitely generated k -algebras to the category of abelian monoids. A morphism $f : A \rightarrow B$ induces a scheme morphism $f^* : \text{Mod}_B(m) \rightarrow \text{Mod}_A(m)$ which is 'restriction of scalars' and thus a map $C_m(f) : C_m(B) \rightarrow C_m(A) : \langle M \rangle \mapsto \langle f^*(M) \rangle$. Let $C(f)$ denote the collection of all these

maps. It clearly preserves the direct sum operation and so defines a homomorphism of abelian monoids.

Proposition 1.1. *If $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ is an exact sequence of A -modules, then $\langle M \rangle = \langle M' \rangle + \langle M'' \rangle$ in $C(A)$.*

Proof. We show that a module isomorphic to $M' \oplus M''$ is in the connected component of M . Let U denote the vector space underlying M' and let V be a complementary subspace to U . Let π_U and π_V denote the projections onto U and V . Define the morphism $\text{Spec } k[t] \rightarrow \text{Mod}_A(U \oplus V)$ by the $A[t]$ -module structure on $(U \oplus V) \otimes k[t] = U[t] \oplus V[t]$ whose scalar multiplication is

$$a \cdot (u + v) = au + t\pi_U(av) + \pi_V(av)$$

and extended in the natural way to make it $k[t]$ -linear. For $t = 1$ we get the module M and for $t = 0$ we get a module isomorphic to $M' \oplus M''$.

Let $M \supset M_1 \supset \dots \supset M_r \supset 0$ be a composition series for M . Repeated use of Proposition 1 shows that

$$\langle M \rangle = \langle M/M_1 \rangle + \langle M_1/M_2 \rangle + \dots + \langle M_r \rangle.$$

Therefore $C(A)$ is generated by the components $\langle L \rangle$ where L is a simple A -module.

Define a module L to be *d-simple* if every module in its component is simple. Thus L is simple and cannot be deformed to a nonsimple module.

Proposition 1.2. *The components $\langle L \rangle$ with L d-simple form the unique minimal set of generators for $C(A)$.*

Proof. If L is *d-simple* then $\langle L \rangle \neq \langle M_1 \rangle + \langle M_2 \rangle$ for $M_1, M_2 \neq 0$. Thus any set of generators must include $\langle L \rangle$. Now every component $\langle M \rangle = \langle L_1 \rangle + \dots + \langle L_r \rangle$ where L_i is simple. If L_i is not *d-simple* then $\langle L_i \rangle = \langle N \rangle + \langle P \rangle$ since $\langle L_i \rangle$ contains a nonsimple module and thus a decomposable module by Proposition 1. Continuing this process one eventually arrives at *d-simple* modules because the dimensions of the summands decrease.

Are the *d-simple* components a free set of generators? In other words, is the decomposition of $\langle M \rangle$ into *d-simple* components unique? Unfortunately I cannot answer this in general but in several cases it is true and there are no counterexamples.

Consider the following conjectures:

Conjecture (1). $C(A)$ is the free monoid on the set of *d-simple* components.

Conjecture (2). $C(A)$ has the cancellation property.

The truth of (1) implies the truth of (2) since a free monoid is isomorphic to $\mathbb{N}^{(S)}$ for some set S and $\mathbb{N}^{(S)}$ has the cancellation property. Conjecture (2), though weaker, is unknown in general and has the same status as (1). It is known to be true in the same special cases as (1) and there are no counterexamples. The truth of (2) would imply that the natural map of $C(A)$ into its associated group is injective.

Example 1.3. Let A be a finite dimensional algebra. Thus $C(A) \cong \mathbb{N}^s$ where s is the number of non-isomorphic simple A -modules. There is a one-to-one correspondence between the connected components of $\text{Mod}_A(m)$ containing k -rational points and the semisimple modules of dimension m . See [5, 3].

Example 1.4. Let A be a commutative algebra and let k be algebraically closed. Then $C(A) \cong \mathbb{N}^s$ where s is the number of factors in the decomposition of A into connected subalgebras, $A = A_1 \times \cdots \times A_s$. See [5, Theorem 2.8].

Note. I have been unable to remove the restriction that k be algebraically closed.

Example 1.5. Let k be algebraically closed with characteristic zero. Let A be the enveloping algebra of a finite dimensional Lie algebra \mathfrak{g} . Then $C(A) \cong \mathbb{N}^{(S)}$ where S is the set of non-isomorphic simple $\mathfrak{g}/\text{rad } \mathfrak{g}$ -modules. Note that S may be an infinite set as it is for $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$. If \mathfrak{g} is solvable then $\text{rad } \mathfrak{g} = \mathfrak{g}$ and we let S consist of the single element that is the zero-module of dimension one so $C(A) \cong \mathbb{N}$ which means the only discrete invariant is the dimension. For $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ there is a simple module in each positive dimension and so $S = \mathbb{N}$. Thus $C(A)$ is the infinite direct sum of a countable number of copies of \mathbb{N} .

Underlying this example is the theorem [5, Theorem 3.1] that every representation of \mathfrak{g} may be deformed to one for which $\text{rad } \mathfrak{g}$ acts trivially as zero. Then one has a representation of $\mathfrak{g}/\text{rad } \mathfrak{g}$ whose isomorphism class is a discrete invariant. The proof of the theorem uses Levi's Theorem and the existence of eigenvectors for representations of solvable Lie algebras. I do not know how to extend the theorem to fields which are not algebraically closed or to characteristic p .

Proposition 1.6. For k -algebras A and B the following holds:

- (i) $C(A \times B) \cong C(A) \times C(B)$.
- (ii) $C(A *_k B) \cong C(A) \times_{\mathbb{N}} C(B)$, fiber product over \mathbb{N} .
- (iii) Let $I \subset A$ be a nilpotent two-sided ideal. Then $C(A) \cong C(A/I)$.

Proof. (i) Every $A \times B$ -module M splits into $M_1 \oplus M_2$ where M_1 is an A -module and M_2 is a B -module. The dimensions of the factors are the same for every other module in the same component as M within $\text{Mod}_{A \times B}(m)$ as can be seen by considering the induced $k \times k$ -modules via $k \times k \rightarrow A \times B$.

- (ii) $A *_k B$ has the universal property that $\text{Hom}_{k\text{-alg}}(A *_k B, E) = \text{Hom}_{k\text{-alg}}(A, E) \times$

$\text{Hom}_{k\text{-alg}}(B, E)$. Let $E = \text{End}_k(k^m)$. Thus an m -dimensional $A *_k B$ -module is a pair made up of an A -module and a B -module each of dimension m . Then $C_m(A *_k B) = C_m(A) \times C_m(B)$ or

$$C(A *_k B) = C(A) \times_{\mathbb{N}} C(B).$$

(iii) Let $\pi : A \rightarrow A/I$, $C(\pi) : C(A/I) \rightarrow C(A)$. We will show $C(\pi)$ is an isomorphism by showing that each A -module M has an A -module N in its component with $I \subset \text{ann}(N)$. Note $C(\pi)$ is an injection so we consider $C(A/I)$ as a submonoid of $C(A)$. Let n be the least integer such that $I^n = 0$.

Lemma 1.7. *If $J \subset A$ is an ideal, $J^2 = 0$, then $C(A/J) \cong C(A)$.*

Proof. Let $A = B \oplus J$, direct sum over k . Let the A -module M have structure map $\varrho : A \rightarrow \text{End}_k(k^m)$ and define

$$\varrho_t : A \rightarrow \text{End}_k(k^m) : (b + u) \mapsto \varrho(b) + t\varrho(u).$$

It is easy to check that ϱ_t is an algebra homomorphism. (To be precise, one must extend ϱ_t to $A \otimes k[t] \rightarrow \text{End}_k(k^m) \otimes k[t] = \text{End}_{k[t]}(k[t]^m)$ to get an $A \otimes k[t]$ -module.) Then ϱ_0 is an A -module with $\varrho_0|_J = 0$ so ϱ_0 is actually an A/J -module. Thus $C(A/J) \rightarrow C(A)$ is onto.

For the rest of the proof of (iii) proceed inductively to get

$$C(A/I^{j+1}) = C((A/I^j)/(I^j/I^{j+1})).$$

Thus $C(A/I^n) = C(A)$.

Remark. The modules of $A \otimes_k B$ do not have such a nice description as those for $A *_k B$ since a representation $\varrho : A \otimes_k B \rightarrow \text{End}_k(k^m)$ consists of a pair σ and τ which are representations of A and B , respectively, and whose images in $\text{End}_k(k^m)$ commute. $\text{Mod}_{A \otimes B}(m)$ is a closed subscheme of $\text{Mod}_A(m) \times \text{Mod}_B(m)$ consisting of the pairs (ϱ, σ) such that $[\varrho(a_i), \sigma(b_j)] = 0$ as a_i and b_j range over sets of generators for A and B . The connected components of this subscheme are not easy to describe in terms of the components of $\text{Mod}_A(m) \times \text{Mod}_B(m)$.

Let us first consider a polynomial algebra B , either commutative or non-commutative. Thus $B = k[x_1, \dots, x_n]$ or $B = k\{x_1, \dots, x_n\}$. The homomorphism $\varepsilon : B \rightarrow k : x_i \mapsto 0$ gives a one-dimensional B -module M_1 whose class $\langle M_1 \rangle$ generates $C(B)$. It is easy to see that any B -module with structure map $\sigma : B \rightarrow \text{End}_k(k^m)$ may be connected to $\oplus^m M_1$ by the family $\sigma_t : B \rightarrow \text{End}_k(k^m) \otimes k[t] : x_i \mapsto t\sigma(x_i)$. We have σ_0 as the structure map for $\oplus^m M_1$.

The algebra homomorphisms $i : A \rightarrow A \otimes B : a \mapsto a \otimes 1$ and $\pi : A \otimes B \rightarrow A : a \otimes b \mapsto \varepsilon(b)a$ induce monoid homomorphisms $C(i) : C(A \otimes B) \rightarrow C(A)$ and $C(\pi) : C(A) \rightarrow C(A \otimes B)$.

Proposition 1.8. For $B = k[x_1, \dots, x_n]$ or $B = k\{x_1, \dots, x_n\}$, $C(A \otimes B) \cong C(A)$ and $C(i)^{-1} = C(\pi)$.

Proof. An $A \otimes B$ -module is determined by an A -module structure $\varrho : A \rightarrow \text{End}_k(k^m)$ and $\sigma : B \rightarrow \text{End}_k(k^m)$ such that $\sigma(x_i)$ commutes with $\varrho(A)$. We can deform σ to σ_0 with $\sigma_0(x_i) = 0$ and thereby get an $A \otimes B$ -module which comes from the A -module structure ϱ by making x_i act as zero. Thus $C(i)$ is surjective. But $C(i)$ is injective since $C(i) \circ C(\pi) = \text{id}_{C(A)}$.

A commutative k -algebra B is *formally smooth* if for every k -algebra homomorphism $\phi : B \rightarrow E/I$ where E is commutative and $I^2 = 0$ there exists a lifting $\tilde{\phi} : B \rightarrow E$ such that $\phi = \pi \circ \tilde{\phi}$, π being the projection $E \rightarrow E/I$.

Now extend this definition to an arbitrary k -algebra B by requiring that $\tilde{\phi}$ exist for any k -algebra homomorphism $\phi : B \rightarrow E/I$, $I^2 = 0$ and E not necessarily commutative.

Note that if B is formally smooth and I is a nilpotent ideal then $\phi : B \rightarrow E/I$ may be lifted to $\tilde{\phi} : B \rightarrow E$ by a sequence of lifts $\phi_2 : B \rightarrow E/I^2$, $\phi_3 : B \rightarrow E/I^3, \dots$. Since $I^n = 0$ we use $\tilde{\phi} = \phi_n$.

To generalize 1.8 we make the following assumptions:

- (i) B is formally smooth in the category of k -algebras.
- (ii) $C(B)$ is generated by a single class $\langle M_1 \rangle$ in dimension one.
- (iii) k is algebraically closed.

Now if B is commutative we replace (i) and (ii):

- (i') B is formally smooth in the category of commutative k -algebras.
- (ii') B is connected.
- (iii) k is algebraically closed.

Together (ii') and (iii) imply (ii). See Example 1.3.

Theorem 1.9. With either set of assumptions above, $C(A \otimes B) \cong C(A)$.

Proof. Let $\varepsilon : B \rightarrow k$ be the structure map of M_1 or any k -point of B if B is commutative. In either case $\langle M_1 \rangle$ is a generator for $C(A)$. Let $\varrho : A \otimes B \rightarrow \text{End}_k(k^m)$ be the structure map of an $A \otimes B$ -module. We will deform ϱ to map σ with $\sigma(a \otimes b) = \varepsilon(b)\sigma(a)$. This will show that $A \otimes B \rightarrow A : a \otimes b \mapsto \varepsilon(b)a$ induces an isomorphism $C(A) \rightarrow C(A \otimes B)$.

Let $f = \varrho|_B$ and let $R = f(B)$. R is a finite dimensional algebra, so $R = S + N$, the semidirect sum of a separable subalgebra S and the radical N . Suppose $N^n = 0$. We will deform f to g in the scheme of algebra homomorphisms from B to R , where $g = \pi_S \circ f$ and $\pi_S : R \rightarrow S$ is the projection onto S . Let $U = f^{-1}(N)$ and $B_0 = f^{-1}(S)$. Suppose we have constructed $f_i : B \rightarrow R$ such that $f_i(U) \subset N^i$ and $f_i = f$ on B_0 . Let $\varrho_{i+1} : R \rightarrow R/N^{i+1}$ and $\pi_i : R/N^{i+1} \rightarrow R/N^i$ be the projections. Define the family

$$\phi : B \rightarrow R/N^{i+1} \otimes k[t] : b + u \mapsto f(b) + t\varrho_{i+1}(f(u))$$

where $b \in B_0$ and $u \in U$. Note that ϕ is actually a homomorphism because $\varrho_{i+1}(f_i(u)) \subset \text{Ker } \pi_i$, a square zero ideal. Now lift ϕ to a map $\tilde{\phi}: B \rightarrow R \otimes k[t]$ which can be done since B is formally smooth. We let $f_{i+1} = \tilde{\phi}|_{t=0}$ and we see that $f_{i+1}(U) \subset N^{i+1}$. Continuing in this way we arrive at $g = f_n$ with $g(U) \subset N^n = 0$ and $g = f$ on R_0 . Thus $g = \pi_S \circ f$.

The algebra S is a product of simple algebras each of which is a matrix algebra over k since k is algebraically closed. As S contains the identity of $\text{End}_k(k^m)$ we have

$$\text{End}_k(k^{m_1}) \times \dots \times \text{End}_k(k^{m_v}) \cong S \subset \text{End}_k(k^m)$$

with $\sum m_i = m$. Thus the B -module defined by g is a direct sum of B -modules of dimensions m_1, \dots, m_v . Each of those modules has a structure map $B \rightarrow \text{End}_k(k^{m_i})$. By assumption (ii) or (ii'), that structure map is in the same component as $B \rightarrow \text{End}_k(k^{m_i}): b \mapsto \varepsilon(b)I$. We simultaneously deform g to the direct sum to get the structure map $B \rightarrow \text{End}_k(k^m): b \mapsto \varepsilon(b)I_m$. This deformation is taking place inside S which commutes with $\varrho(A)$ so we have an actual deformation of ϱ to σ of the form $\sigma(a \otimes b) = \varepsilon(b)\sigma(a)$.

Remark. Theorem 1.9 asserts that a Künneth homomorphism $C(A) \otimes C(B) \rightarrow C(A \otimes_k B)$ is an isomorphism under the conditions that B is formally smooth, $C(B) \cong \mathbb{N}$, and k is algebraically closed. Since $C(B) \cong \mathbb{N}$, the left hand side is isomorphic to $C(A)$. The Künneth homomorphism maps $\langle M \rangle \otimes \langle N \rangle$ to $\langle M \otimes_k N \rangle$ where $M \otimes_k N$ where $M \otimes_k N$ is an $A \otimes_k B$ -module in the usual way. The inverse constructed in the proof of Theorem 1.9 maps the class $\langle M \rangle$ to $\langle {}_A M \rangle \otimes \langle M_1 \rangle$ where ${}_A M$ means M as an A -module and M_1 is the one-dimensional B -module whose class generates $C(B)$.

The results above – Theorem 1.9, Propositions 1.6 and 1.8 – also hold with D in place of C .

In order to count the classes in $C(A)$ define the power series $\kappa_A(t) = \sum_{n=0}^{\infty} v_n t^n$ where $v_n = \text{cardinality of } C_n(A)$ or the number of connected components with k -rational points in $\text{Mod}_A(k^n)$.

Proposition 1.10. (i) $\kappa_{A \times B}(t) = \kappa_A(t)\kappa_B(t)$.

(ii) $\kappa_A(t) = \kappa_{A/I}(t)$ when I is a nilpotent two-sided ideal.

Proof. These follow from Proposition 1.6.

Example 1.11. If A is finite dimensional then $\kappa_A(t) = \kappa_{A/N}(t)$ where N is the radical of A . Since A/N is semisimple, $A/N \cong S_1 \times \dots \times S_s$ with each S_i simple. Now $C(S_i)$ has one generator in dimension m_i . Thus

$$\kappa_{S_i}(t) = 1 + t^{m_i} + t^{2m_i} + \dots = \sum t^{nm_i} = \frac{1}{1 - t^{m_i}}.$$

Thus $\kappa_A(t) = \prod_{i=1}^s 1/(1-t^{m_i})$ where m_1, \dots, m_s are the dimensions of the simple A -modules.

Example 1.12. If A is commutative and k algebraically closed, then [5, Theorem 2.8] shows $\kappa_A(t) = 1/(1-t)^s$ where $A = A_1 \times \dots \times A_s$ and each A_i is connected.

Example 1.13. For $A = U(\mathfrak{sl}(2, \mathbb{C}))$ there is a simple module in each dimension and so v_n is the number of partitions of n . Thus $\kappa_A(t)$ is the classical partition function $\kappa_A(t) = \prod_{i=1}^{\infty} 1/(1-t^i)$.

If Conjecture (1) is true then $C(A) \cong \mathbf{N}^{(S)}$ where S indexes the d -simple components and $\kappa_A(t) = 1/\prod (1-t^{m_s})$ where m_s is the dimension of the corresponding simple module.

1.2. The monoid $H(A)$ and the group $R(A)$

We define two modules M and N to be H -equivalent if there is a sequence of modules $M = M_0, M_1, \dots, M_s = n$ and morphisms $\phi_i: U_i \rightarrow \text{Mod}_A$ where U_i is an affine open subscheme of \mathbf{P}_k^1 and the image of ϕ_i contains M_i and M_{i+1} . Each ϕ_i is a homotopy of modules if we enlarge upon Gersten's definition (see the introduction) to allow rational curves as parameter spaces, not just \mathbf{A}_k^1 . We denote by $H(A)$ the set of H -equivalence classes of Mod_A and make $H(A)$ into an abelian monoid using $\oplus: \text{Mod}_A \times \text{Mod}_A \rightarrow \text{Mod}_A$ which respects H -equivalence. We denote the associated group by $R(A)$.

Then H is a functor from the category of finitely generated k -algebras to the category of abelian monoids and R is a functor to the category of abelian groups. There is a morphism of functors $\Phi: H \rightarrow C$ such that $\Phi_A: H(A) \rightarrow C(A): [M] \mapsto \langle M \rangle$ where $[M]$ is the H -equivalence class of M . Since H -equivalence is finer than C -equivalence this functor is well defined and Φ_A is surjective for every algebra A .

Given an exact sequence $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ we have $[M] = [M'] + [M'']$ since the deformation of M to $M' \oplus M''$ takes place over \mathbf{A}^1 , giving an H -equivalence. Therefore the classes of simple modules form a set of generators for $H(A)$.

Proposition 1.14. (i) $H(A \times B) \cong H(A) \times H(B)$.

(ii) $H(A *_k B) \cong H(A) \times_{\mathbf{N}} H(B)$.

(iii) $H(A/I) \cong H(A)$, I nilpotent two-sided ideal.

Proof. The proofs are the same as for Proposition 1.6.

If A and B are k -algebras, there is a Künneth morphism $H(A) \otimes H(B) \rightarrow H(A \otimes_k B)$ which takes $[M] \otimes [N]$ to $[M \otimes_k N]$ with $M \otimes_k N$ an $A \otimes_k B$ -module whose scalar multiplication is given by $a \otimes b \cdot x \otimes y = ax \otimes by$. If $B = k[x_i]$ or $B = k\{x_i\}$ then $H(B) = \mathbf{N}$ and the Künneth morphism is an isomorphism.

Proposition 1.15. *If $B = k[x_i]$ or $B = k\{x_i\}$ then $H(A) \otimes H(B) \rightarrow H(A \otimes B)$ is an isomorphism.*

Proof. First, $H(B) \cong \mathbb{N}$ since any representation of B may be deformed by a line to the zero-representation $x_i \mapsto 0$. Then any representation of $A \otimes B$ deforms to one with the action of A remaining unchanged while the action of B is the zero-representation. This representation of $A \otimes B$ is the tensor product of the representation restricted to A with the one dimensional zero representation of B .

Theorem 1.16. *Let B satisfy the assumptions of Theorem 1.9. Then $H(A \otimes B) \cong H(A)$.*

Proof. The proof of Theorem 1.9 works again because all the deformations used there are done over \mathbb{A}^1 .

Remark. The results in this section also hold with R in place of H .

1.3. Zero cycles

In this section k is algebraically closed and A is a commutative k -algebra. Let $X = \text{Spec } A$. The group of 0-cycles of X , denoted by $\mathcal{Z}_0(X)$, is the free abelian group on the closed points of X . Using an appropriate definition of ‘rational equivalence’ we show that the group $\mathcal{Z}_0(X) / \mathcal{Z}_0^{\text{rat}}(X)$ is isomorphic to $R(A)$. Thus, $R(A)$ serves as a noncommutative generalization of the Chow group of 0-cycles. Likewise we show that $D(A)$ is isomorphic to the group $\mathcal{Z}_0(X) / \mathcal{Z}_0^{\text{alg}}(X)$, which is the group of 0-cycles modulo algebraic equivalence.

Definition 1.17. Let X be a k -scheme. The group $\mathcal{Z}_p(X)$ of p -cycles on X is the free abelian group on the irreducible subvarieties of X of dimension p (i.e. on the reduced and irreducible subschemes of dimension p).

Note. If X is not reduced, $\mathcal{Z}_p(X) = \mathcal{Z}_p(X_{\text{red}})$ so we may assume X is a variety.

Definition 1.18. The 0-cycle $z \in \mathcal{Z}_0(X)$ is *rationally equivalent to 0* if there exists an open subset $U \subset \mathbb{P}^1$, two points u_1 and u_2 in U , and a 1-cycle $w \in \mathcal{Z}_1(U \times X)$, finite and flat over U , such that $z = w_1 - w_2$ where w_i is the 0-cycle on X defined by projecting the fiber of w at u_i onto X [2].

The group of 0-cycles rationally equivalent to 0 is denoted by $\mathcal{Z}_0^{\text{rat}}(X)$ and the quotient $\mathcal{Z}_0(X) / \mathcal{Z}_0^{\text{rat}}(X)$ by $A_0(X)$. The group $A_0(X)$ is the Chow group of 0-cycles.

If, in the definition of ‘rationally equivalent to 0’, we replace the rational curve U by an arbitrary connected variety T , then the cycle z is said to be ‘algebraically equivalent to 0’. We denote by $\mathcal{Z}_0^{\text{alg}}(X)$ the subgroup of 0-cycles algebraically equivalent to 0.

Now let M be an A -module of dimension n over k . We define a 0-cycle $Z(M) = \sum m_i x_i$ where m_i is the multiplicity of the A -module $\kappa(x_i)$ in a composition series for M . Recall that $\kappa(x_i)$ is the one-dimensional module $A/m(x_i)$ and $m(x_i)$ is the maximal ideal of x_i . Since k is algebraically closed, every simple A -module has dimension one and appears as the residue field of a unique maximal ideal.

Theorem 1.19. *The map $R(A) \rightarrow A_0(X): [M] \rightarrow [Z(M)]$ is an isomorphism.*

Proof. First we must show that the map $\text{Mod}_A \rightarrow \mathcal{X}_0(X): M \rightarrow Z(M)$ respects the R -equivalence classes in Mod_A . Suppose M and N are equivalent in $R(A)$. Then there exists P such that $M \oplus P$ and $N \oplus P$ are in the image of a morphism $\phi: C \rightarrow \text{Mod}_A$ from a rational curve C . By normalizing we may assume C is nonsingular and we may assume it is affine since the points mapped to $M \oplus P$ and $N \oplus P$ lie in an affine subset. Thus we may assume C is open in \mathbb{P}^1 . The morphism $\phi: C \rightarrow \text{Mod}_A$ is an $A \otimes R$ -module on R^N , $N = \dim_k M \oplus P$ and $\text{Spec } R = C$. This gives a module over R which is finite and flat; in fact it is free. Therefore the 1-cycle associated to this module in $\mathcal{X}_1(C \times X)$ is finite and flat over C and shows that $Z(M \oplus P)$ and $Z(N \oplus P)$ are rationally equivalent. Since Z is additive we have $[Z(M \oplus P)] = [Z(M)] \oplus [Z(P)]$ and so $[Z(M)] = [Z(N)]$. This shows the map $R(A) \rightarrow A_0(X)$ is well defined. It is surjective since the 0-cycle $\sum n_i x_i$ is the image of $\sum n_i \kappa(x_i)$.

To show injectivity suppose that $[Z(M)] = [Z(N)]$ in $A_0(X)$. There is a cycle $w \in \mathcal{X}_1(U \times X)$, finite and flat over U , such that $w_a = Z(M)$ and $w_b = Z(N)$ for points $a, b \in U$. Since w is finite and flat over U , w can be written $w = \sum n_i w_i$ where w_i is represented by the irreducible curve $W_i \subset U \times X$, flat and finite over U . Cover U with $\{U_{ij}\}_j$ such that $\mathcal{O}(W_i|U_{ij})$ is a flat $\mathcal{O}(U_{ij})$ module and finitely generated. Each U_{ij} is affine and so is the complement of a finite set of points in \mathbb{A}^1 . Thus $\mathcal{O}(U_{ij})$ is obtained from $k[t] = \mathcal{O}(\mathbb{A}^1)$ by adjoining elements of the form $1/f$. This shows $\mathcal{O}(U_{ij})$ is principal and so $\mathcal{O}(W_i|U_{ij})$ is free. Thus $\mathcal{O}(W_i)$ is projective over $\mathcal{O}(U)$ and hence also free. Let m_i be the rank.

Each curve W_i defines an $A \otimes \mathcal{O}(U)$ -module on $\mathcal{O}(U)^{m_i}$ in the natural way. The sum $\sum n_i w_i$ can be separated into positive and negative parts to define two families of A -modules parameterized by U . These families are given by morphisms $\phi_1, \phi_2: U \rightarrow \text{Mod}_A$. Then we have $M \oplus \phi_2(a) \cong \phi_1(a)$ and $N \oplus \phi_1(b) \cong \phi_2(b)$, so $[M] = [\phi_1(a)] - [\phi_2(a)]$ and $[N] = [\phi_1(b)] - [\phi_2(b)]$ in $R(A)$. Thus $[M] = [N]$ in $R(A)$. This shows the injectivity.

Remark. In the definition of rational equivalence we require finiteness. This prevents the cycle from disappearing by going to infinity. It gives a well-defined degree map on a noncomplete variety X .

Theorem 1.20. *The map $D(A) \rightarrow \mathcal{X}_0(X)/\mathcal{X}_0^{\text{alg}}(X): \langle M \rangle \rightarrow [Z(M)]_{\text{alg}}$ is an isomorphism.*

Proof. The same arguments used in the previous theorem show that this map is well defined and surjective. For injectivity we cannot use the rationality of the parameter space, so we give a different proof which can be used for the injectivity in Theorem 1.19 as well.

Suppose that $Z(M)$ and $Z(N)$ are algebraically equivalent. Let $w \in \mathcal{L}_1(T \times X)$ be a flat, finite cycle over T and $a, b \in T$ such that $w_a = Z(M)$ and $w_b = Z(N)$, and T is a connected k -scheme. Let $w = \sum n_i w_i$ where each w_i is the cycle of an irreducible curve $W_i \subset T \times X$, which is flat and finite over T . By finiteness W_i is affine over T and that together with flatness implies W_i is locally free over T . Cover T with affine neighborhoods T_j so that $W_{ij} = W_i|_{T_j}$ is free over T_j , i.e. $\mathcal{O}(W_{ij})$ is a finite free $\mathcal{O}(T_j)$ -module. Then $\mathcal{O}(W_{ij})$ is a family of A -modules parameterized by T_j since $\mathcal{O}(W_{ij})$ is a quotient of $\mathcal{O}(T_j) \otimes A$, being the coordinate ring of the subscheme $W_{ij} \subset T_j \times A$. Since T is connected and is covered by the open subvariety T_j , all modules in the families $\mathcal{O}(W_{ij})$, for i fixed, are C -equivalent. Thus M and N are C -equivalent and so the map $\langle M \rangle \rightarrow [Z(M)]_{\text{alg}}$ is injective and an isomorphism.

2. Finitely generated modules

2.1. The category \mathcal{M}_A

Let \mathcal{M}_A denote the category of finitely generated left A -modules and assume A is a k -algebra. We define abelian monoids $C(\mathcal{M}_A)$ and $H(\mathcal{M}_A)$ and their associated groups $D(\mathcal{M}_A)$ and $R(\mathcal{M}_A)$ in a manner similar to what was done in Section 1 for finite dimensional modules.

Definition 2.1. Two finitely generated left A -modules M_1 and M_2 are *C-equivalent* if there exists a commutative k -algebra R of finite type and two k -points of R $\varepsilon_1, \varepsilon_2: R \rightarrow k$ and a finitely generated left $A \otimes_k R$ -module E such that E is flat over R and the A -modules $E_1 = E \otimes_{\varepsilon_1} k$ and $E_2 = E \otimes_{\varepsilon_2} k$ are isomorphic over A to M_1 and M_2 , respectively.

Definition 2.2. The modules M_1 and M_2 are *homotopic* or *H-equivalent* if there is a finite sequence of modules $M_1 = N_1, N_2, \dots, N_r = M_2$ where N_i and N_{i+1} are C -equivalent over a k -algebra R_i which is the coordinate ring of a rational curve.

Define the monoid $C(\mathcal{M}_A)$ to consist of the C -equivalence classes of modules in \mathcal{M}_A and define $H(\mathcal{M}_A)$ to be the monoid of H -equivalence classes. Let $D(\mathcal{M}_A)$ and $R(\mathcal{M}_A)$ denote the associated abelian groups.

For an algebra morphism $f: A \rightarrow B$ such that B is a flat A -module there are monoid morphisms $C(f)_*: C(\mathcal{M}_A) \rightarrow C(\mathcal{M}_B)$ and $H(f)_*: H(\mathcal{M}_A) \rightarrow H(\mathcal{M}_B)$ which map the class of M to the class of $M \otimes_A B$.

If $f: A \rightarrow B$ makes B into a finitely generated A -module then there are morphisms

$C(f)^*: C(\mathcal{M}_B) \rightarrow C(\mathcal{M}_A)$ and $H(f)^*: H(\mathcal{M}_B) \rightarrow H(\mathcal{M}_A)$ mapping the class of the B -module N to the class of the A -module ${}_A N$. (${}_A N$ denotes the A -module structure on N obtained by restricting the scalar multiplication to A via f .)

Proposition 2.3. *If $I \subset A$ is a nilpotent ideal and $\pi: A \rightarrow A/I$ is the projection, then $C(\pi)^*$ and $H(\pi)^*$ are isomorphisms.*

Proof. Let f^* denote either map. Then f^* is surjective because any A -module M is homotopic to an A/I -module, i.e. an A -module with I acting as 0. To show f^* is injective let $f^*(M_1) = f^*(M_2)$. Then there is a flat family of A -modules E containing M_1 and M_2 . Now E can be deformed by homotopies to a module E' with $I \otimes R$ acting as 0 (E is an $A \otimes R$ -module). Thus E' is a flat family of $A/I \otimes R$ -modules and so M_1 and M_2 are equivalent as A/I -modules.

Proposition 2.4. *There are isomorphisms $C(\mathcal{M}_A) \oplus C(\mathcal{M}_B) \rightarrow C(\mathcal{M}_{A \times B})$ and $H(\mathcal{M}_A) \oplus H(\mathcal{M}_B) \rightarrow H(\mathcal{M}_{A \times B})$.*

Proof. These isomorphisms are given by $p_1^* + p_2^*$ for $p_1: A \times B \rightarrow A$ and $p_2: A \times B \rightarrow B$.

Proposition 2.5. *There is a natural group homomorphism $\eta: K_0(\mathcal{M}_A) \rightarrow R(\mathcal{M}_A): [M] \rightarrow \langle M \rangle$. Thus K_0 -equivalence is finer than R -equivalence.*

Proof. Suppose $0 \rightarrow M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow 0$ is an exact sequence in \mathcal{M}_A . Then $[M_2] = [M_1] + [M_3]$. But M_2 can be deformed over A^1 to $M_1 \oplus M_3$ and so $\langle M_2 \rangle = \langle M_1 \rangle + \langle M_3 \rangle$. Thus η is well defined.

Proposition 2.6. *If A is left noetherian then $\eta: K_0(\mathcal{M}_A) \rightarrow R(\mathcal{M}_A)$ is an isomorphism.*

Proof. We will show that $\eta^{-1}(\langle M \rangle) = [M]$ gives a well-defined inverse for η . Suppose $\text{Spec } R$ is a nonsingular, rational, affine curve. Then $\text{Spec } R$ is isomorphic to an open subset of A^1 . Thus R is obtained from $k[t]$ by adjoining elements of the form $1/(t-a)$. Then $K_0(\mathcal{M}_A) \cong K_0(\mathcal{M}_{A \otimes R})$. In one direction the isomorphism is given by $[M] \mapsto [M \otimes R]$. The inverse is $[E] \mapsto [E_a] - [E_{(t-a)\text{-tor}}]$ where $a \in \text{Spec } R \subset A^1$. Note that this is the alternating sum $\sum (-1)^i \text{Tor}_i^{A \otimes R}(E, A)$ where A is an $A \otimes R$ -module via $A \otimes R \rightarrow A: \alpha \otimes r \mapsto r\alpha, \bar{r} = r \text{ mod } (t-a)$. Any of the points of $\text{Spec } R$ define the same map since all are inverses of a single map. Thus in $K_0(\mathcal{M}_A)$ we have $[E_a] - [E_{(t-a)\text{-tor}}] = [E_b] - [E_{(t-b)\text{-tor}}]$.

Now suppose E is a flat (over R) family of A -modules. Then $\langle E_a \rangle = \langle E_b \rangle$ by definition of R -equivalence, but also $[E_a] = [E_b]$ since E has no $(t-a)$ -torsion or $(t-b)$ -torsion. Thus $\eta^{-1}(\langle E_a \rangle) = \eta^{-1}(\langle E_b \rangle)$ and so η^{-1} makes sense.

2.2. Notes on projective modules

Let \mathcal{P}_A denote the category of finitely generated, projective left A -modules. Define monoids $C(\mathcal{P}_A)$, $H(\mathcal{P}_A)$ and groups $D(\mathcal{P}_A)$, $R(\mathcal{P}_A)$ just as for the category \mathcal{M}_A . A family of projective A -modules over $\text{Spec } R$ is projective $A \otimes R$ -module E . (It is superfluous to require that E be flat as an R -module. E is a direct summand of $(A \otimes R)^n$. As an R -module $A \otimes R$ is free since A is free over k . Thus E is a projective R -module.)

Proposition 2.7. *If A is left-regular then $K_0(\mathcal{P}_A) \cong R(\mathcal{P}_A)$.*

Proof. Define $\xi: K_0(\mathcal{P}_A) \rightarrow R(\mathcal{P}_A): [P] \rightarrow \langle P \rangle$. Then ξ is well defined in general and not just when A is regular. To define ξ^{-1} we need the regularity of A . Let $\xi^{-1}(\langle P \rangle) = [P]$ and suppose E is a projective $A \otimes R$ -module. By Proposition 2.6 we know that the class in $K_0(\mathcal{M}_A)$ is constant for the modules in the family E . But since A is regular $K_0(\mathcal{P}_A) \cong K_0(\mathcal{M}_A)$ [1, p. 453]. Thus the class in $K_0(\mathcal{P}_A)$ is also constant and we get the inverse map.

References

- [1] H. Bass, Algebraic K -theory (Benjamin, New York, 1968).
- [2] J.-L. Colliot-Thélène and J.-J. Sansuc, La R -équivalence sur les tores, Ann. Sci. Ecole Norm. Sup. 10 (1977) 175–230.
- [3] P. Gabriel, Finite representation type is open, in: Proceedings of the international conference on representations of algebras, Lecture Notes in Math. No. 488 (Springer, Berlin, 1976) 132–155.
- [4] S. Gersten, Homotopy theory of rings, J. Algebra 19 (1971) 396–415.
- [5] K. Morrison, The scheme of finite dimensional representations of an algebra, Pacific J. Math. 91 (1981) 199–218.
- [6] D. Quillen, Higher algebraic K -theory: I, in: Algebraic K -theory, Vol. I, Lecture Notes in Math. No. 341 (Springer, Berlin, 1973) 85–147.