Random Walks with Decreasing Steps

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Abstract

We consider a one-dimensional random walk beginning at the origin and moving $\pm\, c_n$ at time n. If $\sum c_n^2 < \infty$ then with probability one the random walk has a final location. We describe the distribution of the final location and find the cumulants and the moments in terms of the power sums of the step sizes. Some interesting definite integrals involving infinite products of trigonometric functions can be evaluated probabilistically. Similar results hold for random walks with steps chosen uniformly in $[-b_n, b_n]$, assuming that $\sum b_n^2 < \infty$.

1 Harmonic Random Walk

Imagine a random walker who is not only staggering but also tiring. In contrast to a simple random walker who takes steps of equal length but random direction, our harmonic random walker first takes a step of length 1, then a step of length 1/2, then a step of length 1/3, and so on. As in a simple random walk the direction of each step is equally likely to be left as right and independent of the other steps. Assume the walk begins at the origin at time zero. What is the distribution of the location as time goes to infinity?

For the simple random walk the location at time n has a binomial distribution with variance n (assuming steps of unit length), and so there is not a limiting distribution unless rescaling is done as in the Central Limit Theorem in which the location at time n is scaled to units of $\sqrt{n}$ so that the distribution tends to a standard normal distribution. However, the harmonic random walk does have a limiting distribution for the location as time goes to infinity, as do many other random walks with variable steps.

The fundamental probability space Ω consists of countable binary sequences to represent the sequence of left-right choices in a random walk. We identify Ω with the interval $[0, 1]$ by means of the binary representation of numbers in $[0, 1]$ and we use Lebesgue measure on $[0, 1]$ as the probability measure.

Let $c_n \geq 0$ be the length of the step at time n and define random variables X_n on Ω to have the value c_n or $-c_n$ according to whether the nth step is to the right or to the left. The random walk is then the countable sequence of random variables

$$X_1, X_1 + X_2, \ldots, X_1 + X_2 + \cdots + X_n, \ldots$$

and the final location is the random variable S defined by

$$S = \sum_{n=1}^{\infty} X_n.$$

Now in general S converges for some ω in Ω and not for others, but Rademacher proved that the set $\{\omega \in \Omega | S \text{ converges}\}$ has probability one if and only if $\sum c_n^2 < \infty$. Otherwise it has probability zero. In [7] Kac presents a proof due to Paley and Zygmund of Rademacher's theorem. Some years later Kolmogorov proved a more general theorem: an infinite sum $\sum X_n$ of independent random variables converges with probability one if the sum of the variances of the X_n converges. These results are illustrative of another well-known theorem of Kolmogorov, his Zero-One Law. Events (by which we mean measurable subsets) that depend on the tails of the sequences, such as convergence does, must have probabilistic measure zero or one.

The harmonic random walk does have a final location, since $\sum n^{-2}$ converges, and so it makes sense to ask, for example, for the probability that the final location is between 1 and 2.

With MATLAB we simulated 10,000 random walks with 200 steps each and plotted a histogram showing the location at the 200th step.

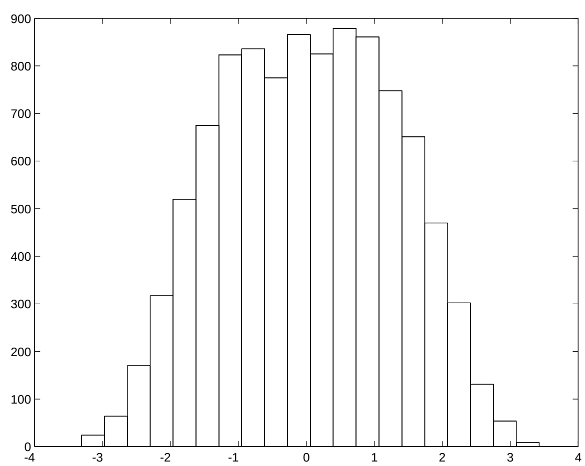

Figure 1. Location at step 200 of 10,000 random walks.

Note the sharp fall-off of the distribution around ± 2.

2 Geometric Random Walks

Let us imagine a random walker who tires much faster than the the harmonic random walker in that his steps are of length $c_n = 2^{-n}$. The final resting place in this case is the sum

$$S = \sum_{n=1}^{\infty} a_n 2^{-n},$$

where $a_n \in \{-1, 1\}$. We see that S is distributed uniformly in the interval $[-1, 1]$ because

$$\begin{aligned} S + 1 &= \sum_{n=1}^{\infty} a_n 2^{-n} + \sum_{n=1}^{\infty} 2^{-n} \\ \frac{1}{2}(S+1) &= \sum_{n=1}^{\infty} \frac{1}{2}(a_n + 1) 2^{-n} \\ &= \sum_{n=1}^{\infty} \omega_n 2^{-n}, \end{aligned}$$

where $\omega_n \in \{0, 1\}$. Thus $(S+1)/2$ is precisely the choice of a point ω in our probability space Ω, which is the unit interval $[0, 1]$. Note that the distribution of S has a sudden drop-off at the ends.

We define a *geometric random walk* to be one whose step size $c_n = \lambda^n$ for a fixed $\lambda > 0$. Of course, if $\lambda > 1$, then the final location S does not converge and for $\lambda = 1$ we have the simple random walk. Therefore we assume that $0 < \lambda < 1$. Since we already understand the case of $\lambda = 1/2$, let us consider the geometric random walk with $\lambda = 1/3$. Then

$$S = \sum_{n=1}^{\infty} a_n 3^{-n}, \ a_n \in \{-1, 1\}.$$

Adding $1/2$ to S is the same as adding 1 to each a_n since $1/2 = \sum_{n=1}^{\infty} 3^{-n}$. Thus,

$$S + 1/2 = \sum_{n=1}^{\infty} t_n 3^{-n}, \ t_n \in \{0, 2\}.$$

This is the ternary representation of a number in $[0, 1]$ that uses only 0 and 2, and we see that $S + 1/2$ ranges over the Cantor set. Thus, S is distributed over the Cantor set constructed from $[-1/2, 1/2]$ by removing the middle thirds. The distribution of S is a measure μ whose cumulative distribution function $F(x)$, defined by $F(x) = \mu([-1/2, x))$, is continuous and non-decreasing with $F(-1/2) = 0$, $F(1/2) = 1$ and $F'(x) = 0$ almost everywhere. The measure μ is singular with respect to Lebesgue measure.

What about other λ? The full story is not yet known. For $0 < \lambda < 1/2$ the distribution of S is a singular measure supported on a (generalized) Cantor set. However, for $1/2 < \lambda < 1$ it is not known exactly which values of λ define random walks for which S has a singular distribution. A strong conjecture is that there are only a countable number for which this happens. This problem has a long history beginning in the 30's when Jessen and Wintner proved that the distribution of S is either singular or absolutely continuous. In 1939 Erdös [3, 4] showed that the distribution is singular for values of λ that are reciprocals of PV numbers. (A Pisot-Vijayaraghavan number is an algebraic integer whose Galois conjugates are all less than one in absolute value.) No other singular λ are known. In 1995 Solomyak [12, 10] showed that almost every λ between 1/2 and 1 gives rise to a distribution for S that is absolutely continuous with respect to Lebesgue measure and that the corresponding density function is in $L^2(\mathbf{R})$. Some explicit values that give absolutely continuous distributions are $\lambda = 2^{-1/k}$ for k a positive integer. (See Example 3.)

3 The Characteristic Function of S

The function ϕ defined by

$$\phi(t) = \mathbf{E}(e^{itS})$$

is the **characteristic function** of S. It is easy to see that

$$\begin{aligned} \phi(t) &= \mathbf{E}(e^{it\sum X_n}) \\ &= \mathbf{E}(\prod e^{itX_n}) \\ &= \prod \mathbf{E}(e^{itX_n}) \\ &= \prod \frac{1}{2}(e^{itc_n} + e^{-itc_n}) \\ &= \prod \cos c_n t. \end{aligned}$$

Let μ be the distribution of S, so that μ is a probability measure on $\mathbf{R}$. Then we also have

$$\phi(t) = \int_{-\infty}^{\infty} e^{itx}\, d\mu(x).$$

Theorem 3.1 *Assume that the sequence $\{c_n\}$ is square-summable. Consider the product $\prod \cos c_n t$. Then the following hold.*

(a) The product converges absolutely for all t.

(b) The product converges uniformly on compact sets.

(c) The function $\phi(t) = \prod \cos c_n t$ is continuous, in fact, uniformly continuous.

(d) The function ϕ has a Fourier transform in the distributional sense and its transform is the measure μ.

Proof (a) and (b) Choose ϵ to satisfy

$$1 - \frac{\epsilon^2}{2} < \cos \epsilon \leq 1.$$

For t in $[a, b]$, and n sufficiently large (say $n > n_0$) $c_n t$ is in $(-\epsilon, \epsilon)$, and so

$$1 - \frac{(c_n t)^2}{2} < \cos c_n t \leq 1.$$

We may as well assume that $0 \leq a < b$ since the product of cosines is an even function. Thus, for $n > n_0$

$$1 - \frac{(c_n b)^2}{2} < \cos c_n t \leq 1$$

and hence the tail of the product satisfies

$$\prod_{n>n_0}^{\infty} \left(1 - \frac{(c_n b)^2}{2}\right) < \prod_{n>n_0}^{\infty} \cos c_n t \leq 1.$$

The product $\prod_{n>n_0}^{\infty} \left(1 - \frac{(c_n b)^2}{2}\right)$ converges because $\frac{(c_n b)^2}{2}$ is summable. This shows that the tail of the product goes to 1 uniformly on $[a, b]$ and therefore the product converges uniformly on $[a, b]$.

(c) Use the Dominated Convergence Theorem.

(d) Follows from the Continuity Theorem. See, for example Breiman [1, Theorem 8.28] or Durrett [2, Chapter 2, (3.4)] or Feller [5, XV.3, Theorem 2]. □

From (a) it follows that the order of the c_n does not matter in the distribution of S. This might seem surprising because rearranging the c_n certainly changes the sequences $\omega \in \Omega$ for which convergence occurs. We may as well assume that the c_n are arranged monotonically: $c_n \geq c_{n+1}$.

Example 3.2 Here is a quick proof of a classic identity. Let $c_n = 2^{-n}$, which gives a distribution for S that is uniform on $[-1, 1]$. The characteristic function of this measure is

$$\begin{aligned} \phi(t) &= \int_{-\infty}^{\infty} e^{itx}\, d\mu(x) \\ &= \frac{1}{2} \int_{-1}^{1} e^{itx}\, dx \\ &= \frac{\sin t}{t}. \end{aligned}$$

Thus,

$$\prod_{n=1}^{\infty} \cos \frac{t}{2^n} = \frac{\sin t}{t}. \qquad (1)$$

□

Example 3.3 Consider the geometric random walk with $\lambda = 1/\sqrt{2}$. Separating the factors of $\phi(t) = \prod \cos \lambda^n t$ according to the parity of the power of λ we see that

$$\phi(t) = \prod_{m=1}^{\infty} \cos \frac{t}{2^m} \prod_{m=1}^{\infty} \cos \frac{\sqrt{2}t}{2^m}.$$

Using the cosine product identity (1) we see that

$$\phi(t) = \frac{\sin t}{t} \frac{\sin \sqrt{2}t}{\sqrt{2}t}.$$

The first factor is the characteristic function for the uniform distribution on $[-1, 1]$ and the second factor is the characteristic function for the uniform distribution on $[-\sqrt{2}, \sqrt{2}]$. The convolution of the two uniform distributions is a "triangular" distribution whose density is shown in Figure 2. Similar reasoning shows that for $\lambda = 2^{-1/k}$, k a positive integer, the characteristic function is a product of k factors, each of which is the characteristic function of a uniform distribution. The density is a the convolution of k uniform densities on the intervals $[-1, 1]$, $[-2^{1/k}, 2^{1/k}]$,..., $[-2^{(k-1)/k}, 2^{(k-1)/k}]$. One can show the density is $k - 2$ times continuously differentiable and that the derivative of order $k - 1$ is piecewise continuous. These values of λ give distributions that are absolutely continuous with respect to Lebesgue measure. Figure 2 shows the density for $\lambda = 1/\sqrt[3]{2}$.

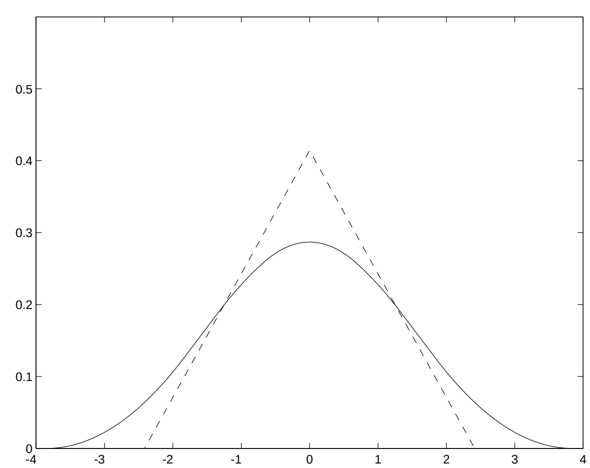

Figure 2. Densities for geometric random walks with $\lambda = \frac{1}{\sqrt{2}}$ (dashed) and $\lambda = \frac{1}{\sqrt[3]{2}}$ (solid).

□

Via the Fourier transform, properties of ϕ are reflected in properties of the distribution of S. If ϕ is in $L^1(\mathbf{R})$, then $\mu = f(x)dx$ with f continuous and vanishing at infinity. And if ϕ is in $L^2(\mathbf{R})$, then $\mu = f(x)dx$ with f in $L^2(\mathbf{R})$. In either case, the distribution of S is absolutely continuous with respect to Lebesgue measure. Solomyak's result shows that the product $\prod \cos \lambda^n t$ is extraordinarily sensitive to changes in λ. For almost every λ in $(1/2, 1)$, the product $\prod \cos \lambda^n t$ is in $L^2(\mathbf{R})$, but every once in a while it is not in L^2 and not in L^1. What tends to make μ singular is gaps in the distribution of S and that can occur if the step sizes c_n go to 0 too rapidly. A good example is the Cantor measure that comes from $c_n = 3^{-n}$. If the sequence c_n is square-summable but not summable, then for each $s \in \mathbf{R}$ there are sign sequences $\omega \in \Omega$ for which $s = \sum \omega_n c_n$. For a given $s > 0$ we construct ω by setting $\omega_n = 1$, $1 \leq n \leq k$, where k is the smallest index for which $\sum_n^k c_n > s$. Then set ω_n to -1 until the partial sum drops below s. Continue like this with the partial sums going over and under s. Since the c_n are going to 0, the partial sums approach s. One can also see that there are a multitude of ω for which the sum is s because the same construction can be applied to the sequence after any arbitrary initial segment of ω. This gives a heuristic argument that the distribution of S is absolutely continuous with respect to Lebesgue measure when c_n is square-summable but not summable, but a solid proof of this is not available. A rigorous treatment beyond the level of this article appears in Offner [9]. With further assumptions about the growth of the c_n he shows that the distribution of S is very smooth and decays rapidly at infinity. Offner's results do include the case of the random harmonic series $c_n = 1/n$, and so we know that the final position of the harmonic random walk is smoothly distributed and decays rapidly as shown by the histogram in Figure 1 and by the plot of the density in Figure 3.

Although the distribution of S can be singular with respect to Lebesgue measure, it cannot be so singular as to have any point masses or atoms. That is, the measure of a singleton set is always 0. This means S is known as a continuous random variable, the term "continuous" coming from the fact that the cumulative distribution function (defined later in §4) of S is continuous.

Theorem 3.4 *Let μ be the distribution of the random variable S. Then $\mu(\{s\}) = 0$ for every $s \in \mathbf{R}$.*

Proof Let c_{n_k} be a subsequence with the property that $c_{n_{k+1}} < (1/2)c_{n_k}$. Define S_1 to be the random variable

$$S_1 = \sum_k X_{n_k}$$

and define $S_2 = S - S_1$. Let μ, μ_1, and μ_2 be the distributions of S, S_1, and S_2. Then S_1 and S_2 are independent and so $\mu = \mu_1 * \mu_2$. In order for μ to have a point mass at s, there must be points s_1 and s_2 at which μ_1 and μ_2 have mass with $s = s_1 + s_2$. But μ_1 has no point masses because the possible values that occur for S_1 occur for just one sequence of signs on account of the restriction on the size of the steps. □

4 The probability that S lies in an interval

We will derive an integral formula for the probability that S lies in a given interval and then apply it to give simple justifications for some rather complicated integrals.

Theorem 4.1 *For* $S = \sum X_n$

$$\mathbf{P}(0 < S < a) = \frac{1}{\pi}\int_0^\infty \frac{\sin at}{t} \prod_{n=1}^{\infty} \cos c_n t \, dt.$$

Proof For all the probabilities involving S it does not matter whether the intervals are open, closed, or half-open, because the distribution of S has no point masses. We will find a formula for $\mathbf{P}(-a < S < a)$ and from that we will easily obtain others. Let χ denote the characteristic function of the interval $(-a, a)$. Then

$$\mathbf{P}(-a < S < a) = \int_{-\infty}^{\infty} \chi(x) \, d\mu(x).$$

The Fourier transform of χ is

$$\hat{\chi}(t) = \frac{1}{\pi}\frac{\sin at}{t}.$$

We write χ in terms of its Fourier transform

$$\chi(x) = \int_{-\infty}^{\infty} \frac{1}{\pi}\frac{\sin at}{t} e^{itx} \, dt.$$

Then

$$\begin{aligned}
\mathbf{P}(-a < S < a) &= \frac{1}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\sin at}{t} e^{itx} \, dt \, d\mu(x) \\
&= \frac{1}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\sin at}{t} e^{itx} \, d\mu(x) \, dt \\
&= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin at}{t} \phi(t) \, dt \\
&= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin at}{t} \prod_{n=1}^{\infty} \cos c_n t \, dt.
\end{aligned}$$

Since the integrand is even,

$$\mathbf{P}(-a < S < a) = \frac{2}{\pi}\int_0^\infty \frac{\sin at}{t} \prod_{n=1}^{\infty} \cos c_n t \, dt.$$

And since the distribution of S is symmetric about the origin,

$$\mathbf{P}(0 < S < a) = \frac{1}{\pi}\int_0^\infty \frac{\sin at}{t} \prod_{n=1}^{\infty} \cos c_n t \, dt.$$

□

Corollary 4.2

$$\mathbf{P}(a < S < b) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin bt - \sin at}{t} \prod_{n=1}^{\infty} \cos c_n t \, dt.$$

$$\mathbf{P}(0 \leq a < S < b) = \frac{1}{\pi} \int_{0}^{\infty} \frac{\sin bt - \sin at}{t} \prod_{n=1}^{\infty} \cos c_n t \, dt.$$

The **cumulative distribution function** of a random variable X is defined by

$$F(x) = \mathbf{P}(X \leq x), \quad -\infty < x < \infty.$$

Corollary 4.3 *Let $F(x)$ be the distribution function of S. Then for $x > 0$*

$$\begin{aligned} F(x) &= \frac{1}{2} + \frac{1}{\pi} \int_{0}^{\infty} \frac{\sin at}{t} \prod_{n=1}^{\infty} \cos c_n t \, dt \\ F(-x) &= 1 - F(x). \end{aligned}$$

Proof Use the symmetry of the distribution of S and the theorem. □

Because the distribution of S has no point masses, the function F is continuous. In case the measure μ is absolutely continuous with respect to Lebesgue measure, the derivative of F is the density associated to S and to μ. In general, however, there is the Lebesgue decomposition $\mu = \mu_s + \mu_a$ with μ_s singular and μ_a absolutely continuous with respect to Lebesgue measure. Then $F'(x)$ is the density of μ_a. In the last corollary we sweep all the analytic difficulties and unknown behavior under the rug.

Corollary 4.4 *If the distribution of S is absolutely continuous with respect to Lebesgue measure and if it is allowable to differentiate under the integral sign, then for $x > 0$*

$$f(x) = \frac{1}{\pi} \int_{0}^{\infty} \cos xt \prod_{n=1}^{\infty} \cos c_n t \, dt.$$

Example 4.5 For the harmonic random walk the results of [9] show that the distribution of S is absolutely continuous and that the density is in fact

$$f(x) = \frac{1}{\pi} \int_{0}^{\infty} \cos xt \prod_{n=1}^{\infty} \cos \frac{t}{n} \, dt.$$

See Figure 3 for a plot of this density. The integration was done numerically. Compare with the histogram in Figure 1.

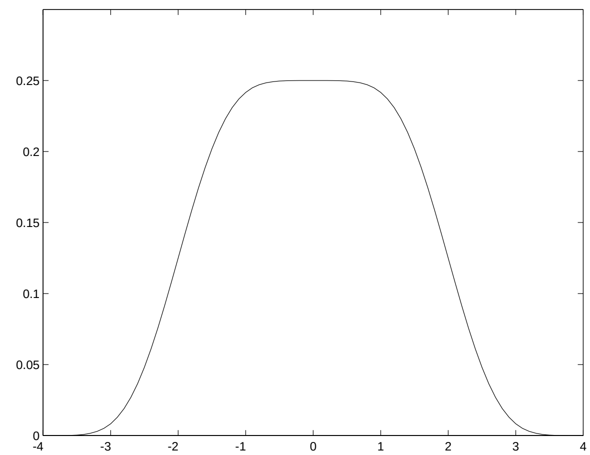

Figure 3. Graph of the density $f(x)$ for the harmonic random walk.

□

We can use these results to evaluate some rather difficult integrals.

Theorem 4.6 *Suppose that* $\sum c_n$*is finite and that* $a > \sum c_n$*. Then*

$$\int_0^\infty \frac{\sin at}{t} \prod_{n=1}^\infty \cos c_n t \, dt = \frac{\pi}{2}.$$

Proof It is impossible for S to be larger than the sum $\sum c_n$. Therefore $\mathbf{P}(0 < S < a) = 1/2$ and our result follows from the previous proposition. □

Corollary 4.7 *If* $\sum c_n^2 < \infty$*, then*

$$\lim_{a \to \infty} \int_0^\infty \frac{\sin at}{t} \prod_{n=1}^\infty \cos c_n t \, dt = \frac{\pi}{2}.$$

5 Sums of Uniform Random Variables

We consider a fatigued random walker whose nth step is chosen uniformly from the interval $[-b_n, b_n]$. We can assume that b_n is a positive sequence tending monotonically to 0. We define the final position to be the sum

$$R = \sum_{n=1}^\infty Y_n$$

where Y_n is uniform on $[-b_n, b_n]$. Since the variance of Y_n is $b_n^2/3$, it follows that R converges if and only if $\{b_n\}$ is square-summable. It is easy to compute the characteristic function of Y_n to see that

$$\mathbf{E}(e^{itY_n}) = \frac{\sin b_n t}{b_n t}$$

and so the characteristic function of R is

$$\mathbf{E}(e^{itR}) = \prod_{n=1}^{\infty} \frac{\sin b_n t}{b_n t}.$$

Following the same steps as for the random variable S in the previous section we see that

Theorem 5.1

$$\mathbf{P}(0 < R < a) = \frac{1}{\pi} \int_0^{\infty} \frac{\sin at}{t} \prod_{n=1}^{\infty} \frac{\sin b_n t}{b_n t}$$

From this it follows that

Corollary 5.2 *If the b_n are summable and $a > \sum b_n$, then*

$$\int_0^{\infty} \frac{\sin at}{t} \prod_{n=1}^{\infty} \frac{\sin b_n t}{b_n t} = \frac{\pi}{2}.$$

Theorem 5.3 (Gradshteyn and Ryzhik [6, 3.746 and 3.836]) *If $a > c_1 + \cdots + c_m + b_1 + \cdots + b_n$, then*

$$\int_0^{\infty} \frac{\sin at}{t} \cos c_1 t \cdots \cos c_m t \frac{\sin b_1 t}{t} \cdots \frac{b_n t}{t}\, dt = b_1 b_2 \cdots b_n \frac{\pi}{2}$$

Proof We have a probabilistic proof for this. In fact, it proves a more general result by considering the sum $S + R = \sum X_n + \sum Y_n$, with the assumption that c_n and b_n are summable and that $a \geq \sum c_n + \sum b_n$. Then the characteristic function of $S + R$ is the product

$$\mathbf{E}(e^{it(S+R)}) = \prod_{n=1}^{\infty} \cos c_n t \prod_{n=1}^{\infty} \frac{\sin b_n t}{b_n t}$$

and

$$\mathbf{P}(0 < S + R < a) = \int_0^{\infty} \frac{1}{\pi} \frac{\sin at}{t} \prod_{n=1}^{\infty} \cos c_n t \prod_{n=1}^{\infty} \frac{\sin b_n t}{b_n t}$$

but with our assumptions $Pr(0 < S + R < a) = 1/2$. This shows that

$$\frac{1}{\pi} \int_0^{\infty} \frac{\sin at}{t} \prod_{n=1}^{\infty} \cos c_n t \prod_{n=1}^{\infty} \frac{\sin b_n t}{b_n t} = \frac{1}{2}$$

from which obtain the integral in Gradshteyn and Ryzhik by specializing to the case of finite sequences. □

Let us look again at the harmonic random walk with $c_n = 1/n$ and characteristic function $\phi(t) = \prod \cos c_n t$. Each positive integer n has the form $n = 2^k m$ where m is odd. Thus, we can factor $\phi(t)$ as

$$\phi(t) = \prod_{m \text{ odd}} \prod_{k=0}^{\infty} \cos \frac{t}{2^k m}.$$

Recall the identity (1), which gives the identity

$$\prod_{k=0}^{\infty} \cos \frac{t}{2^k m} = \frac{\sin \frac{2}{m} t}{\frac{2}{m} t}.$$

Thus,

$$\phi(t) = \prod_{n=1}^{\infty} \frac{\sin b_n t}{b_n t} \tag{2}$$

where $b_n = 2/(2n-1)$. This means that ϕ is the characteristic function of $R = \sum Y_n$, with Y_n uniform in $[-2/(2n-1), 2/(2n-1)]$.

In an earlier paper [8] we noticed that numerical integration of the density

$$f(x) = \frac{1}{\pi} \int_0^{\infty} \cos xt \prod_{n=1}^{\infty} \cos \frac{t}{n} \, dt \tag{3}$$

gave a value for $f(0)$ suspiciously close to $1/4$ and a value for $f(2)$ suspiciously close to $1/8$ and we wondered whether those are the actual values. We will show that they are not the actual values and explain why they are so close. Using (2) and (3) we have

$$f(0) = \frac{1}{\pi} \int_0^{\infty} \frac{\sin 2t}{2t} \prod_{m \geq 3, \text{ odd}}^{\infty} \frac{\sin \frac{2}{m} t}{\frac{2}{m} t} \, dt.$$

Now the right side is $(1/2)\mathbf{P}(0 < R < 2)$ where $R = Y_3 + Y_5 + Y_7 + \cdots$ and Y_m is uniform on $[-2/m, 2/m]$. But it is very unlikely that the value can be as large as 2 because the distribution of R is the same as for the harmonic random walk with all steps omitted for n a power of 2. Even using the largest possible steps it requires the partial sum $1/3 + 1/5 + 1/7 + \cdots + 1/30$ with 25 steps to surpass 2. This gives an idea that it is exceedingly difficult for the sum to be as much as 2.

To see why $f(2)$ is so close to $1/8$ but not quite equal to it, proceed as follows:

$$\begin{aligned} f(2) &= \frac{1}{\pi} \int_0^{\infty} \cos 2t \prod_{n \geq 1} \cos \frac{t}{n} \, dt \\ &= \frac{1}{\pi} \int_0^{\infty} \cos 2t \frac{\sin 2t}{2t} \prod_{m \geq 3, \text{ odd}}^{\infty} \frac{\sin \frac{2}{m} t}{\frac{2}{m} t} \, dt \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{4\pi}\int_0^\infty \frac{\sin 4t}{t} \prod_{m\geq 3,\ \text{odd}}^{\infty} \frac{\sin\frac{2}{m}t}{\frac{2}{m}t}\,dt \\
&= \frac{1}{4}\mathbf{P}(0<R<4),
\end{aligned}
$$

where R is the random variable just defined.

6 The moments of S

In this section we compute the moments of S using cumulants. Let X be a random variable with characteristic function $\phi_X(t) = \mathbf{E}(e^{itX})$. Expand $\log\phi_X(t)$ in a power series

$$
\log\phi_X(t) = \kappa_1(it) + \ldots + \kappa_n\frac{(it)^n}{n!} + \ldots
$$

The coefficient κ_n is called the nth **cumulant** of X. Cumulants have the virtue of being additive for a sum of independent random variables, whereas the moments are not. Furthermore, the moments can be gotten from the cumulants. For us the following is the most important example, because the cumulants of S are sums of the cumulants of the independent steps. Let X be the random variable that is $\pm a$ with equal probability. Then $\phi_X(t) = \cos at$ and the cumulants are gotten from the series of $\log\cos at$.

Lemma 6.1

$$
\log\cos t = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}B_{2n}2^{2n}(2^{2n}-1)}{2n(2n)!}t^{2n}
$$

Proof Of course, this is ancient history, but to derive it from the beginning one starts with the series defining the Bernoulli numbers

$$
\frac{t}{e^t-1} = \sum_{n=0}^{\infty}\frac{B_n}{n!}t^n. \tag{4}
$$

Also,

$$
\begin{aligned}
\frac{t}{e^t-1}+\frac{t}{2} &= \frac{t(e^t+1)}{2(e^t-1)} \\
&= \frac{te^{-t/2}(e^{t/2}+e^{-t/2})}{2e^{-t/2}(e^{t/2}-e^{-t/2})} \\
&= \frac{t\cosh t/2}{2\sinh t/2} \\
&= \frac{t}{2}\coth\frac{t}{2}
\end{aligned}
$$

With this identity and the power series (4) we also use the identity $\cot t = i \coth it$ to obtain

$$t \cot t = \sum_{n=0}^{\infty} \frac{B_{2n}(-4)^n t^{2n}}{(2n)!}.$$

Next, we use the identity

$$\tan t = \cot t - 2\cot 2t$$

to obtain the series

$$\tan t = \sum_{n=1}^{\infty} \frac{B_{2n}(-4)^n(1-4^n)t^{2n-1}}{(2n)!}.$$

Integrating this we arrive at

$$\log\cos t = \sum_{n=1}^{\infty} \frac{B_{2n}(-4)^n(1-4^n)t^{2n}}{2n(2n)!}$$

from which the lemma follows. □

Lemma 6.2 *Let X be the random variable having value $\pm a$ with equal probability. The odd cumulants of X are 0 and the even cumulants are given by*

$$\kappa_{2n} = \frac{2^{2n-1}(2^{2n}-1)B_{2n}a^{2n}}{n}$$

or by the equivalent

$$\kappa_{2n} = \frac{(-1)^{n-1}(2^{2n}-1)(2n)!\zeta(2n)a^{2n}}{n\pi^{2n}}$$

where $\zeta(x)$ is the Riemann zeta function.

Proof The first formula results from rewriting the series for $\log\cos at$ in the previous lemma in powers of it. The second formula follows from the first using the classical formula

$$B_{2n} = \frac{(-1)^{n-1}(2n)!\zeta(2n)}{2^{2n-1}\pi^{2n}}.$$

This formula can be found in Whittaker and Watson [13, §13.151, but beware the difference in notation for the Bernoulli numbers]. □

Lemma 6.3 *The odd cumulants of S are 0 and the even cumulants are given by*

$$\kappa_{2n} = \frac{2^{2n-1}(2^{2n}-1)B_{2n}}{n} p_{2n}$$

where $p_{2n} = \sum_k c_k^{2n}$.

Proof This follows from the additivity of the cumulants of the individual steps X_k, which are given by Lemma 6.2. □

Lemma 6.4 (Moments from the cumulants) *The nth moment m_n is related to the cumulants κ_j by the formula*

$$\frac{m_n}{n!} = \sum_{\alpha \in \mathcal{P}(n)} \prod_j \left(\frac{\kappa_j}{j!}\right)^{\alpha_j} \frac{1}{\alpha_j!}.$$

In this formula $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a partition of n where α_j counts the number of j's so that $n = \sum j\alpha_j$.

Proof We have

$$\log \mathbf{E}(e^{itX}) = \sum_{n=0}^{\infty} \frac{\kappa_n (it)^n}{n!}$$

and

$$\mathbf{E}(e^{itX}) = \sum_{n=0}^{\infty} \frac{m_n (it)^n}{n!}$$

from which it follows that

$$\exp\left(\sum_{n=0}^{\infty} \frac{\kappa_n (it)^n}{n!}\right) = \sum_{n=0}^{\infty} \frac{m_n (it)^n}{n!}$$

and hence that

$$\prod_{n=0}^{\infty} \exp\left(\frac{\kappa_n (it)^n}{n!}\right) = \sum_{n=0}^{\infty} \frac{m_n (it)^n}{n!}.$$

Now let

$$\lambda_n = \frac{i^n \kappa_n}{n!} \text{ and } \nu_n = \frac{i^n m_n}{n!}$$

so that we are considering

$$\prod_{n=0}^{\infty} \exp(\lambda_n t^n) = \sum_{n=0}^{\infty} \nu_n t^n.$$

Expanding the exponentials on the left gives

$$\prod_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(\lambda_n t^n)^k}{k!} = \sum_{n=0}^{\infty} \nu_n t^n.$$

From this we see that ν_n only depends on $\lambda_1, \ldots, \lambda_n$ and that ν_n is the sum

$$\nu_n = \sum \frac{\lambda_1^{\alpha_1}}{\alpha_1!} \frac{\lambda_2^{\alpha_2}}{\alpha_2!} \cdots \frac{\lambda_n^{\alpha_n}}{\alpha_n!}$$

where the sum is over n-tuples $(\alpha_1, \ldots, \alpha_n)$ such that $\alpha_1 + 2\alpha_2 + \cdots + n\alpha_n = n$. Of course, such an n-tuple is a partition of n. The lemma follows easily from the definition of λ_n and ν_n. $\square$

In order to apply this lemma to S the sum over partitions need only be taken over even partitions into even parts. Let $\mathcal{P}_{ev}(2n)$ denote the partitions of $2n$ into even parts. An element of $\mathcal{P}_{ev}(2n)$ is a $2n$-tuple $(\alpha_1, \ldots, \alpha_{2n})$ of non-negative integers such that $\alpha_{2j+1} = 0$ and $\sum_{j=1}^{n} \alpha_{2j}(2j) = 2n$.

Theorem 6.5 *Let The odd moments of S are 0 and the even moments are given by*

$$m_{2n} = (2n)! \sum_{\alpha \in \mathcal{P}_{ev}(n)} \prod_{j} \left(\frac{2^{2j-1}(2^{2j}-1)B_{2j}p_{2j}}{2j(2j)!} \right)^{\alpha_{2j}} \frac{1}{\alpha_{2j}!}.$$

Proof Substitute the formula for the cumulants of S from Lemma 3 into the formula for the moments in Lemma 4. □

The formula just derived is quite complicated. As an aid to understanding it, some of the low order moments are

$$\begin{aligned} m_2 &= p_2 \\ m_4 &= 3p_2^2 - 2p_4 \\ m_6 &= 15p_2^3 - 30p_2p_4 + 16p_6. \end{aligned}$$

Example 6.6 For the harmonic random walk the moments are expressible in terms of the Riemann zeta function because the power sums are values of the zeta function

$$p_{2k} = \sum_{n=1}^{\infty} \frac{1}{n^{2k}} = \zeta(2k).$$

Therefore

$$\begin{aligned} m_2 &= \zeta(2) \\ m_4 &= 3\zeta(2)^2 - 2\zeta(4) \end{aligned}$$

Recalling that $\zeta(2) = \pi^2/6$ and $\zeta(4) = \pi^4/90$,

$$\begin{aligned} m_2 &= \frac{\pi^2}{6} \\ m_4 &= \frac{11}{180}\pi^4 \end{aligned}$$

Example 6.7 The distribution for the harmonic random walk looks approximately normal, and one may wonder whether it is possible for any of our fatigued random walks to have a normally distributed final location.

The normal distribution with mean 0 and variance σ^2 has vanishing odd moments and even moments M_{2n} given by

$$M_{2n} = (2n-1)(2n-3)\cdots(3)(1)\sigma^{2n}.$$

One can look up the moments or find M_{2n+2} in terms of M_{2n} using integration by parts. Now we can see that it is impossible for the distribution of $S = \sum X_n$ to be normal by comparing fourth moments. For the normal distribution the fourth moment is $M_4 = 3\sigma^4$, but for S the fourth moment is $m_4 = 3p_2^2 - 2p_4$, which is $3\sigma^4 - 2p_4$, and these clearly are not equal.

Even allowing random walks that are a sum of X_n and Y_n as in §5, a similar argument shows the final location cannot be normally distributed. □

7 The Distribution of $|S|$

How far does the fatigued random walker travel before he rests? We do not care whether he ends up to the right or to the left of the origin and so it is the distribution of $|S|$ that concerns us. In particular we might want to know the average of $|S|$.

Let μ be the distribution (measure) of S, F the cumulative distribution function, and if S has a density with respect to Lebesgue measure let f be that density. Likewise let ν, G, and g be the corresponding objects for $|S|$. We have the following relationships.

Theorem 7.1

(a) For $0 \leq a \leq b$, $\nu((a, b)) = 2\mu((a, b))$.

(b) For $x \geq 0$, $G(x) = 2F(x) - 1$.

(c) For $x \geq 0$, $g(x) = 2f(x)$.

Proof Use the symmetry of the distribution of S. □

Thus, if we know the distribution of S visually, in the sense of having a plot of its density, then we "know" the distribution of $|S|$.

Theorem 7.2 *Let ϕ and ψ be the characteristic functions of S and $|S|$. Let $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote the Fourier transform and inverse transform. Let $\chi_{(0,\infty)}$ be function that is 1 for $x > 0$ and 0 for $x \leq 0$. Then*

$$\psi = 2\mathcal{F}^{-1}(\chi_{(0,\infty)}\mathcal{F}(\phi)).$$

Proof We know that $\mathcal{F}(\phi) = \mu$ and $\mathcal{F}(\psi) = \nu$ and $\nu = 2\chi_{(0,\infty)}\mu$. □

Corollary 7.3 *Let h be the measure*

$$h(t) = \frac{i}{t} + \pi\delta(t).$$

Then

$$\psi = \frac{1}{\pi}(h * \phi).$$

Proof One knows (or computes) that

$$\mathcal{F}(h) = \chi_{(0,\infty)}$$

and then use

$$\mathcal{F}^{-1}(\hat{h}\hat{\phi}) = \frac{1}{2\pi}(h * \phi).$$

□

Corollary 7.4

$$\psi(t) = \frac{i}{\pi}\int_{-\infty}^{\infty} \frac{\phi(t-\tau)}{\tau}\, d\tau + \phi(t).$$

From this we can develop a formula for the expectation of $|S|$.

Theorem 7.5

$$\mathbf{E}(|S|) = \frac{2}{\pi}\int_0^{\infty} \sum_{n=1}^{\infty} \frac{c_n \sin c_n\tau}{\tau} \prod_{m\neq n} \cos c_m\tau\, d\tau.$$

Proof Expand $\psi(t) = \mathbf{E}(e^{it|S|})$ as a power series in t to see that $\psi'(0)/i = \mathbf{E}(|S|)$. Using the corollary above we see that

$$\begin{aligned} \mathbf{E}(|S|) &= \frac{d}{dt}\left(\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\phi(t-\tau)}{\tau}\, d\tau + \frac{\phi(t)}{i}\right)\Bigg|_{t=0} \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\phi'(-\tau)}{\tau}\, d\tau + \frac{\phi'(0)}{i} \end{aligned}$$

Since ϕ is an even function we have $\phi'(0) = 0$. Differentiating we find that

$$\phi'(t) = \sum_n -c_n \sin c_n t \prod_{m\neq n} \cos c_m t.$$

Therefore

$$\mathbf{E}(|S|) = \frac{1}{\pi}\int_{-\infty}^{\infty} \sum_n \frac{c_n \sin c_n\tau}{\tau} \prod_{m\neq n} \cos c_m\tau\, d\tau.$$

The integrand is an even function and so we can integrate from 0 to ∞ to conclude the proof. □

An alternative formula is available for the expectation of $|S|$.

Theorem 7.6

$$E(|S|) = 2\int_0^{\infty} (1 - F(x))\, dx.$$

Proof We will prove this under the additional assumption that the distribution of S (and hence that of $|S|$) is absolutely continuous with respect to Lebesgue measure. This means, of course, that

$$F(x) = \int_{-\infty}^{x} f(u)\,du.$$

(For a proof and discussion in the general case and a formula for all the moments see Feller [5, V.6]. It requires integration by parts for Lebesgue-Stieltjes integrals.) Then

$$\begin{aligned} E(|S|) &= \int_0^\infty xg(x)\,dx \\ &= 2\int_0^\infty xf(x)\,dx \\ &= 2\int_0^\infty xF'(x)\,dx. \end{aligned}$$

Now integrate by parts using $F(x) - 1$ as the antiderivative of $F'(x)$.

$$\begin{aligned} E(|S|) &= 2x(F(x)-1)|_0^\infty - 2\int_0^\infty (F(x)-1)\,dx \\ &= 2\int_0^\infty (1 - F(x))\,dx. \end{aligned}$$

□

This theorem gives a nice geometric picture of the average as twice the area above the graph of $F(x)$ and below 1 and to the right of $x = 0$, but we have not found a compact formula that tells us the expectation of $|S|$ in terms of the c_n. So we know the even moments of $|S|$, since they are the same as the even moments of S, in terms of the c_n and the power sums, but we do not know the odd moments in the same way.

We note that Jensen's Inequality gives an upper bound for $\mathbf{E}(|S|)$

$$\mathbf{E}(|S|) = \mathbf{E}(\sqrt{S^2}) \leq \sqrt{\mathbf{E}(S^2)} = \sqrt{\sum {c_n}^2}.$$

Example 7.7 For the harmonic random walk we resort to numerical integration to approximate $\mathbf{E}(|S|)$. Using the same values for the density $f(x)$ used for the plot in Figure 3 we estimate

$$\mathbf{E}(|S|) = 2\int_0^\infty xf(x)\,dx \approx 1.0758.$$

Jensen's inequality gives an upper bound

$$\sqrt{\mathbf{E}(S^2)} = \sqrt{\frac{\pi^2}{6}} = 1.2825\ldots.$$

A simulation of 10,000 random sums (partial sums up to $n = 200$) gives an estimate of

$$\mathbf{E}(|S|) \approx 1.0761$$

and the same random sums give

$$\mathbf{E}(S^2) \approx 1.6403,$$

which compares favorably with the exact value

$$\mathbf{E}(S^2) = \frac{\pi^2}{6} = 1.6449\ldots.$$

□

References

[1] L. Breiman. *Probability*, Addison-Wesley, Reading, Mass., 1968. Republished by SIAM, Philadelphia, 1992.

[2] R. Durrett. *Probability: Theory and Examples*, Brooks/Cole, Belmont, Calif., 1991.

[3] P. Erdös. On a family of symmetric Bernoulli convolutions. *Amer. J. Math.*, 61: 974–975, 1939.

[4] P. Erdös. On the smoothness properties of Bernoulli convolutions. *Amer. J. Math.*, 62: 180–186, 1940.

[5] W. Feller. *An Introduction to Probability Theory and Its Applications*, vol. II, second edition. John Wiley & Sons, New York, 1971.

[6] I. S. Gradshteyn and I. M. Ryzhik. *Table of Integrals, Series, and Products*, fifth edition. A. Jeffrey, ed. Academic Press, San Diego, 1994.

[7] M. Kac. *Statistical Independence in Probability, Analysis and Number Theory.* Carus Monographs, no. 12. Mathematical Association of America, Washington, D.C., 1959.

[8] K. E. Morrison. Cosine product identities, Fourier transforms, and probability. *Amer. Math. Monthly*, 102:716–724, 1995.

[9] C. D. Offner. Zeros and growth of entire functions of order 1 and maximal type with an application to the random signs problem. *Math. Zeit.*, 175:189–217, 1980.

[10] Y. Peres and B. Solomyak. Absolute continuity of Bernoulli convolutions, a simple proof. *Math. Res. Lett.*, 3:231–239, 1996.

[11] H. Rademacher. Einige Sätze über Reihen allgemeinen Orthogonalfunktionen. *Math. Ann.*, 87:112–138, 1922.

[12] B. Solomyak. On the random series $\sum \pm\lambda^n$ (an Erdös problem). *Ann. Math.*, 142:611–625, 1995.

[13] E. T. Whittaker and G. N. Watson. *A Course of Modern Analysis*, fourth edition. Cambridge University Press, Cambridge, 1927.