THE SCHEME OF FINITE DIMENSIONAL REPRESENTATIONS OF AN ALGEBRA

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For a finitely generated k-algebra A and a finite dimension-al k-vector space M the representations of A on M form an affine k-scheme $\mathrm{Mod}_A(M)$. Of particular interest for this scheme are the connected components, the irreducible components, and the open and closed orbits under the natural action of the general linear group $\mathrm{Aut}_k(M)$, since the orbits are the equiva-lence classes of representations. The connected components are known for a finite dimensional algebra A. In this paper we characterize the connected components when A is com-mutative or an enveloping algebra of a Lie algebra in chara-cteristic zero. For the algebra $k[x, y]/(x, y)^2$ we describe the open orbits and the irreducible components. Finally, we ex-amine the connection with the theory of deformations of al-gebra representations.

Introduction. For almost any k-algebra A it is an impossible task to classify all finite dimensional A-modules. There are some standard exceptions such as the finite dimensional semisimple algebras, $A = k[x]/(x^m)$, $A = k[x, y]/(x, y)^2$, and $A = k[x]$. Even for the polynomial algebra in two indeterminates, $k[x, y]$, Gelfand and Ponomarev have shown that the classification problem is as difficult as classifying the modules over the free algebra $k\{x_1, \cdots, x_m\}$. (See [7] for details.)

If we restrict our attention to modules of a given dimension, say n, then to classify the n-dimensional A-modules is to classify the orbits of $\mathrm{Aut}_k(k^n) = \mathrm{GL}(n)$ acting on the space of n-dimensional A-module structures. If we take A to be a finitely generated k-algebra then this space of module structures is an affine k-scheme, which we call $\mathrm{Mod}_A(M)$ where $M = k^n$. Although we cannot hope to determine the orbits in most cases, we may be able to describe coarser features of $\mathrm{Mod}_A(M)$ such as the irreducible components and the connected components. It is surprising that even for the con-nected components there is not a complete answer for an arbitrary finitely generated k-algebra, while the irreducible components are not at all understood. Only for one interesting algebra, namely $A = k[x, y]/(x, y)^2$, have the irreducible components been determined, and for this it is essential to know the classification of the finite dimensional A-modules. (In § 5 we recall the description of the indecomposable modules from [9], determine all open orbits, and from them describe the irreducible components as found by Flanigan

and Donald [5].)

More is known about the connected components of $\mathrm{Mod}_A(M)$. Assume k is algebraically closed in order to simplify the discussion. If A is finite dimensional over k, then two module structures are in the same connected component if and only if they have isomorphic Jordan-Hölder factors [6]. Therefore the connected components correspond to the semisimple modules. This is an effective criterion, too, because for each simple A-module L the multiplicity of L in the composition series for an A-module structure ρ is given by the rank of $\rho(e)$ where e is a minimal idempotent such that $L \cong Ae/Ne$ and N is the radical of A. Thus, if ρ is given as a set of matrices satisfying the relations of A, then one can compute $rk\ \rho(e)$ for an orthogonal set of minimal idempotents and thereby determine when two module structures are in the same component.

In §3 and §4 we also describe the connected components when A is a finitely generated commutative algebra and when A is the enveloping algebra of a finite dimensional Lie algebra in characteristic zero. These are both infinite dimensional examples, but they do not point to any obvious conjectures that would generalize the results.

One may view $\mathrm{Mod}_A(M)$ as a generalization of $\mathrm{Spec}\,A$ for non-commutative algebras. In fact, for A commutative and k algebraically closed $\mathrm{Mod}_A(k) = \mathrm{Spec}\,A$. A better generalization would be the quotient $\mathrm{Mod}_A(M)/\mathrm{Aut}_k(M)$ but it is not a scheme. There is, however, a categorical quotient X whose points are the isomorphism classes of semisimple A-modules as shown in the work of Procesi [14, 4.1]. The projection $\pi\colon \mathrm{Mod}_A(M) \to X$ maps two module structures to the same point if and only if they have isomorphic Jordan-Hölder factors. Thus we see that when A is finite dimensional, X is the space of connected components of $\mathrm{Mod}_A(M)$. For the special case $A = k[x]$ see Mumford and Suominen [12] and Byrnes and Gauger [2].

In §5 we examine the structure of $\mathrm{Mod}_A(M)$ for the three dimensional algebra $k[x, y]/(x, y)^2$ and describe the open orbits and irreducible components. In §6 we translate the language of deformation theory of modules in the style of [4 and 8] to the geometric setting of $\mathrm{Mod}_A(M)$. Several results from [8] about rigidity are proved by constructing corresponding subschemes of $\mathrm{Mod}_A(M)$.

Throughout this article we use the language of [3].

1. **The scheme $\mathrm{Mod}_A(M)$.** Let k be a commutative ring with unit, A an associative k-algebra and M a k-module. The functor $\mathrm{Mod}_A(M)$ from the category of commutative k-algebras to the category of sets is defined by

$\mathrm{Mod}_A(M)(R) =$ the set of $A \otimes R$-module structures on $M \otimes R$.

This is also the set of R-algebra homomorphisms from $A \otimes R$ to $\mathrm{End}_R(M \otimes R)$ or the set of representations of the R-algebra $A \otimes R$ on the R-module $M \otimes R$.

THEOREM 1.1. *If A is a finitely generated k-algebra and M is a projective k-module of finite type, then $\mathrm{Mod}_A(M)$ is representable by an affine k-scheme.*

If in addition A is finitely presented or k is noetherian, then $\mathrm{Mod}_A(M)$ is an affine k-scheme of finite type.

Proof. Pick a set of generators $a_1, \cdots, a_n$ for A. Let $I = (f_j)_{j \in J}$ be the ideal of relations with generators f_j. For each $j \in J$ define a morphism of functors

$$\tilde{f}_j \colon \mathrm{End}_k(M) \times \cdots \times \mathrm{End}_k(M) \longrightarrow \mathrm{End}_k(M)$$
$$(u_1, \cdots, u_n) \longmapsto f_j(u_1, \cdots, u_n) \, .$$

The functor $\mathrm{End}_k(M)$ is given by $\mathrm{End}_k(M)(R) = \mathrm{End}_R(M \otimes R)$ and is representable [3, II, §1, 2.4]. Now the zero subfunctor $Z \subset \mathrm{End}_k(M)$ is closed and affine, so that $\tilde{f}_j^{-1}(Z)$ is closed and affine. Therefore the functor $\bigcap_{j \in J} \tilde{f}_j^{-1}(Z)$ is closed and affine. The R-points of this functor are clearly n-tuples of R-module endomorphisms which define R-algebra homomorphisms from $A \otimes R$ to $\mathrm{End}_R(M \otimes R)$ by mapping $a_i \otimes 1$ to u_i.

For the finiteness conditions, if A is finitely presented then we can take J to be a finite set. If k is noetherian then the coordinate ring of $\mathrm{End}_k(M)$, which is $S(M^* \otimes M)$, is also noetherian and so the ideal defining $\mathrm{Mod}_A(M)$ in $\mathrm{End}_k(M)^n$ is finitely generated. ☐

With the hypotheses of the theorem $A \mapsto \mathrm{Mod}_A(M)$ is a contravariant functor from the category of finitely generated k-algebras to the category of (affine) k-schemes. An algebra morphism $f \colon A \to B$ induces the morphism of functors $f^* \colon \mathrm{Mod}_B(M) \to \mathrm{Mod}_A(M)$ which is "restriction of scalars".

PROPOSITION 1.2. *Under the hypotheses of 1.1 let $I \subset A$ be a two-sided ideal. Then $\mathrm{Mod}_{A/I}(M) \subset \mathrm{Mod}_A(M)$ is a closed subscheme.*

Proof. Let $\{e_j\}_{j \in J}$ be a set of generators for I. For each $j \in J$ define Z_j to be the closed subscheme whose R-points are the $A \otimes R$-module structures ρ such that $\rho(e_j \otimes 1_R) = 0$ in $\mathrm{End}_R(M \otimes R)$. Then $\mathrm{Mod}_{A/I}(M) = \bigcap_{j \in J} Z_j$.

Let $\mathrm{Aut}_k(M)(R) = \{\phi \colon M \otimes R \to M \otimes R \,|\, \phi$ is an R-module auto-

morphism}. There is a natural action of the k-group scheme $\mathrm{Aut}_k(M)$ on $\mathrm{Mod}_A(M)$ defined for each R by

$$\gamma_R\colon \mathrm{Aut}_k(M)(R) \times \mathrm{Mod}_A(M)(R) \longrightarrow \mathrm{Mod}_A(M)(R)$$

$$\phi, \rho \longmapsto \phi \cdot \rho$$

where $(\phi \cdot \rho)(\alpha) = \phi \circ \rho(\alpha) \circ \phi^{-1}$ for $\alpha \in A \otimes R$. Therefore $\phi \cdot \rho$ and ρ are isomorphic $A \otimes R$-modules.

2. **Topological properties.** We assume that k is a field so that the orbits of γ are subschemes. Let ρ be a point in $\mathrm{Mod}_A(M)(k)$ and let X_ρ be its orbit. Recall that as a functor X_ρ is the sheafification of the functor which assigns to R the set of $A \otimes R$-module structures on $M \otimes R$ isomorphic to ρ_R, where ρ_R is the natural extension of ρ. [3, III, § 3, 5.4]

THEOREM 2.1. *The tangent space at ρ of $\mathrm{Mod}_A(M)$ is the vector space of linear maps $\nu\colon A \to \mathrm{End}_k(M)$ satisfying $\nu(a_1 a_2) = \rho(a_1)\nu(a_2) + \nu(a_1)\rho(a_2)$ for all a_1, a_2 in A. This is the space of one-cocycles of A with values in the two-sided A-module $\mathrm{End}_k(M)$ under the structure map ρ.*

Proof. Let $k[\varepsilon] = k[x]/(x^2)$ be the algebra of dual numbers. A tangent vector at ρ is an element $\tilde{\rho}$ of $\mathrm{Mod}_A(M)(k[\varepsilon])$ lying over ρ, i.e., such that $\mathrm{Mod}_A(M)(\pi)(\tilde{\rho}) = \rho$, where $\pi\colon k[\varepsilon] \to k$ is the homomorphism defined by $\pi(\varepsilon) = 0$. Thus, $\tilde{\rho}$ has the form $\tilde{\rho}(a + \varepsilon b) = \rho(a) + \varepsilon(\rho(b) + \nu(a))$ for some k-linear map $\nu\colon A \to \mathrm{End}_k(M)$. Direct calculation shows that $\tilde{\rho}$ is a homomorphism iff ν satisfies the indicated cocycles condition.

THEOREM 2.2. *The tangent space at ρ of X_ρ is the vector space of linear maps $\nu\colon A \to \mathrm{End}_k(M)$ such that for some $f \in \mathrm{End}_k(M)$ $\nu(a) = [f, \rho(a)]$. This is the space of one-coboundaries of A with values in $\mathrm{End}_k(M)$ under the structure map ρ.*

Proof. The tangent space at I of $\mathrm{Aut}_k(M)$ can be identified with $\mathrm{End}_k(M)$ so that $f \in \mathrm{End}_k(M)$ determines the $k[\varepsilon]$-point $I + \varepsilon f$ with $(I + \varepsilon f)(x + \varepsilon y) = x + \varepsilon(f(x) + y)$. Then $\gamma_\rho(I + \varepsilon f)(a + \varepsilon b) = (I + \varepsilon f) \circ \rho(a + \varepsilon b) \circ (I + \varepsilon f)^{-1}$. Using $(I + \varepsilon f)^{-1} = I - \varepsilon f$, we see that

$$\gamma_\rho(I + \varepsilon f)(a + \varepsilon b) = \rho(a) + \varepsilon(\rho(b) + [f, \rho(a)]) \,. \qquad \square$$

The last two theorems show that the normal space at ρ of X_ρ is isomorphic to $H^1(A, \mathrm{End}_k(M), \rho)$, the first Hochschild cohomology group of A with values in $\mathrm{End}_k(M)$ via the structure map ρ. Let

M_ρ denote M with the A-module structure ρ. We have the following isomorphism:

PROPOSITION 2.3. $H^1(A, \mathrm{End}_k(M), \rho) \cong \mathrm{Ext}^1_A(M_\rho, M_\rho)$.

Proof. Let $\nu: A \to \mathrm{End}_k(M)$ be a one-cocycle for ρ and define an extension

$$0 \longrightarrow M \overset{i}{\longrightarrow} M \otimes k[\varepsilon] \overset{p}{\longrightarrow} M \longrightarrow 0$$

$i(x) = \varepsilon x$, $p(x + \varepsilon y) = x$, and A acts on $M \otimes k[\varepsilon]$ by $a(x + \varepsilon y) = \rho(a)(x) + \varepsilon[\nu(a)(x) + \rho(a)(y)]$. It is easy to check that a coboundary determines a trivial extension.

In the other direction, given an extension

$$0 \longrightarrow M \overset{i}{\longrightarrow} W \overset{p}{\longrightarrow} M \longrightarrow 0$$

we pick a splitting over k of W as $M \oplus \varepsilon M$, so that $i(x) = \varepsilon x$, $p(x + \varepsilon y) = x$. In order for these to be A-module homomorphisms A must act on $M \oplus \varepsilon M$ with $a \in A$ determining an endomorphism of the form $x + \varepsilon y \mapsto \rho(a)(x) + \varepsilon(\nu(a)(x) + \rho(a)(y))$ and ν must be a cocycle. Trivial extensions give rise to coboundaries.

THEOREM 2.4.   *The orbit X_ρ of an A-module structure ρ is an open subscheme of $\mathrm{Mod}_A(M)$ if and only if $H^1(A, \mathrm{End}_k(M), \rho) = 0$.*

Proof. For k algebraically closed it was proved by Gabriel in [6, 1.2]. Therefore assume k is not algebraically closed.

If X_ρ is open then the tangent spaces of X_ρ and $\mathrm{Mod}_A(M)$ are equal at ρ and so $H^1(A, \mathrm{End}_k(M), \rho) = 0$.

Conversely, assume $H^1(A, \mathrm{End}_k(M), \rho) = 0$. Consider the scheme $\mathrm{Mod}_{\bar{A}}(\bar{M})$ where $\bar{A} = A \otimes \bar{k}$, $\bar{M} = M \otimes \bar{k}$ and $\bar{k}$ is an algebraic closure of k. Now $\mathrm{Mod}_{\bar{A}}(\bar{M}) = \mathrm{Mod}_A(M) \times \mathrm{Spec}\ \bar{k}$ and the orbit subscheme of $\rho_{\bar{k}}$ in $\mathrm{Mod}_{\bar{A}}(\bar{M})$ is $X \times \mathrm{Spec}\ \bar{k}$. The projection

$$\mathrm{Mod}_A(M) \times \mathrm{Spec}\ \bar{k} \longrightarrow \mathrm{Mod}_A(M)$$

is open and closed so we need to show that $X_\rho \times \mathrm{Spec}\ \bar{k}$ is open. This follows from the case that the base field is algebraically closed because

$$H^1(\bar{A}, \mathrm{End}_k(\bar{M}), \rho_{\bar{k}}) = H^1(A, \mathrm{End}_k(M), \rho) \otimes \bar{k} = 0 \ .$$

THEOREM 2.5.   *The orbit X_ρ is a closed subscheme if and only if ρ is a semisimple module.*

Proof. (i) In [1, 12.6, p. 559] Artin proved that X_ρ closed implies ρ is semisimple. We give Gabriel's argument [6, 1.3]. The idea is to deform the module M (under ρ) to a module isomorphic to soc $(M) \oplus M/\mathrm{soc}\,(M)$, where soc (M) denotes the socle of M, which is the sum of the simple submodules of M. Then we continue this process with $M/\mathrm{soc}\,(M)$ and so on, until we arrive at a semisimple module, which must happen since the socle is never zero and M has finite dimension.

Let V be a complement to $S = \mathrm{soc}\,(M)$ in M. Define

$$\phi_t\colon S \oplus V \longrightarrow S \oplus V\colon (s,\,v) \longmapsto (ts,\,v)$$

and for every $a \in A$, $\rho_t(a) = \phi_t \circ \rho(a) \circ \phi_t^{-1}$. If we let $t = 0$ then we get a module structure isomorphic to $S \oplus M/S$. This module is in the closure of the orbit X_ρ. Therefore if X_ρ is closed then $M \cong$ soc $(M) \oplus M/\mathrm{soc}\,(M)$, and so soc $(M) = M$ which means M is semisimple (under ρ). Continuing the deformation process with $M/\mathrm{soc}(M)$ shows that in the closure of X_ρ are the semisimple modules with the same Jordan-Hölder factors as ρ.

(ii) For the converse assume ρ is semisimple. We may consider M as a module over the finite dimensional algebra $B = A/\ker \rho$. Since ρ is semisimple, rad $(B) = 0$ and B is a semisimple algebra. All B-modules are projective so that $\mathrm{Ext}_B^1(N, N) = 0$ for all B-modules N. Therefore all orbits of $\mathrm{Mod}_B(M)$ are open in $\mathrm{Mod}_B(M)$. It follows that each orbit must be closed, since each is the complement of the others. By Proposition 1.2 $\mathrm{Mod}_B(M)$ is a closed subscheme of $\mathrm{Mod}_A(M)$, and we have just shown that X_ρ is closed in $\mathrm{Mod}_B(M)$. Therefore X_ρ is closed in $\mathrm{Mod}_A(M)$.

THEOREM 2.6. *If A is finite dimensional and $A/\mathrm{rad}\,(A)$ is separable, then two module structures are in the same connected component if and only if they have isomorphic Jordan-Hölder factors.*

Proof. Let $N = \mathrm{rad}\,(A)$. By the Wedderburn-Malcev Theorem there is a subalgebra S of A which is isomorphic to A/N. Using the injection $i\colon A/N \to A$ which maps A/N onto S, we get a scheme morphism

$$i^*\colon \mathrm{Mod}_A(M) \longrightarrow \mathrm{Mod}_{A/N}(M)\,.$$

The morphism i^* just considers an A-module as an A/N-module. If two module structures are in the same component of $\mathrm{Mod}_A(M)$, then their images under i^* must be in the same component of $\mathrm{Mod}_{A/N}(M)$ or, equivalently, they must lie in the same orbit of $\mathrm{Mod}_{A/N}(M)$ since

A/N is semisimple and the orbits are the connected components. It is easy to see that $i^*(\rho)$ is a semisimple module with the same Jordan-Hölder factors as ρ, since a composition series for ρ is also a composition series for $i^*(\rho)$.

Conversely, if two modules have the same Jordan-Hölder factors then the closures of their orbits intersect and so their orbits must lie in the same connected component. $\qquad\square$

We would like to generalize this theorem to infinite dimensional algebras but it is not true as it stands. For example, let $A = k[x]$. Then $\mathrm{Mod}_A(M)$ is isomorphic to $\mathrm{End}_k(M)$, a connected scheme. There are, however, many nonisomorphic semisimple $k[x]$-modules of the same dimension.

If A is a commutative, finitely generated algebra over an algebraically closed field, then we can describe the connected components of $\mathrm{Mod}_A(M)$. The proof depends upon the Nullstellensatz. Recall that a commutative ring R is said to be *connected* if it has no idempotents other than 0 and 1. Equivalently, the topological space $\mathrm{Spec}\,R$ is connected.

THEOREM 2.7. *If A is a connected, commutative, finitely generated algebra over the algebraically closed field k, then $\mathrm{Mod}_A(M)$ is connected.*

Proof. Fix a basis $e_1, \cdots, e_n$ for M. Define the map

$$f: \mathrm{Mod}_A(\langle e_1 \rangle) \times \cdots \times \mathrm{Mod}_A(\langle e_n \rangle) \longrightarrow \mathrm{Mod}_A(M):$$
$$(\sigma_1, \cdots \sigma_n) \longmapsto \sigma_1 \oplus \cdots \oplus \sigma_n$$

whose image is connected since $\mathrm{Mod}_A(\langle e_i \rangle) \approx \mathrm{Spec}\,A$. The closure of every orbit X_ρ meets the image of f because X_ρ contains a module structure σ which has a composition series $M_0 \subset M_1 \subset \cdots \subset M_n$, $M_i = \langle e_1, \cdots, e_i \rangle$, since simple A-modules are all of dimension one by the Nullstellensatz. Then σ deforms to a semisimple module $M_1 \oplus M_2/M_1 \oplus \cdots \oplus M_n/M_{n-1}$ compatible with the splitting $\langle e_1 \rangle \oplus \cdots \oplus \langle e_n \rangle$. This semisimple module is in the image of f and therefore $\mathrm{Mod}_A(M)$ is connected. $\qquad\square$

Recall that a module E over the product algebra $A_1 \times \cdots \times A_s$ splits into submodules $E_1, \cdots, E_s$ such that $E = E_1 \oplus \cdots \oplus E_s$ and each E_i is an A_i-module, and for $a = (a_1, \cdots, a_s)$ and $x = x_1 + \cdots + x_s$ we have $ax = a_1 x_1 + \cdots + a_s x_s$.

THEOREM 2.8. *Let A be a finitely generated, commutative k-algebra, k algebraically closed. Let $A = A_1 \times \cdots \times A_s$ where A_i is*

connected, and let ρ and σ be two A-module structures on M. Then ρ and σ are in the same connected component if and only if dim $M_i =$ dim M_i' *where M_i and M_i' are the A_i-modules in the decompositions of ρ and σ.*

Proof. Let $B = k \times \cdots \times k$, s factors. We have the algebra morphism $i \colon B \to A$ and the scheme morphism "restriction of scalars" $i^* \colon \mathrm{Mod}_A(M) \to \mathrm{Mod}_B(M)$. If ρ and σ are in the same component then $i^*(\rho)$ and $i^*(\sigma)$ are in the same component of $\mathrm{Mod}_B(M)$. Since B is semisimple $i^*(\rho)$ and $i^*(\sigma)$ are isomorphic B-modules. The isomorphism type of a B-module is determined by the dimensions of the s subspaces that occur in its decomposition. (Some of the subspaces may be 0.) The decomposition of $i^*(\rho)$ coincides with that of ρ and similarly for $i^*(\sigma)$. Therefore dim $M_i =$ dim M_i'.

Conversely, if dim $M_i =$ dim M_i' for $1 \leq i \leq s$, then there is a module structure τ in the orbit of σ having the same decomposition as ρ. That is, the underlying subspaces are the same. Then ρ and τ are in the image of the scheme map

$$S \colon \mathrm{Mod}_{A_1}(M_1) \times \cdots \times \mathrm{Mod}_{A_s}(M_s) \longrightarrow \mathrm{Mod}_A(M)$$

which forms the direct sum of the modules over the product of the algebras. Since each of the schemes $\mathrm{Mod}_{A_i}(M_i)$ is connected by the previous theorem, the product is connected and the image of S is connected. Therefore ρ and τ are in the same connected component, and so ρ and σ are in the same component. $\qquad\square$

In Theorem 2.9 the assumption of commutativity for A is essential as the next section will show. Enveloping algebras of Lie algebras are connected—they have no nontrivial idempotents. They even have skew fields of fractions. However, there are several connected components in $\mathrm{Mod}_{U(L)}(M)$ unless L is solvable.

Theorems 2.6 and 2.8 both characterize the connected components in terms of the components of $\mathrm{Mod}_B(M)$ for a subalgebra B. In the finite dimensional case B is a subalgebra isomorphic to A/N. In the commutative case B is $k \times \cdots \times k$, s factors. In both cases B is a maximal separable subalgebra, and B is unique up to conjugation by a unit. Can one use this observation to describe the connected components of $\mathrm{Mod}_A(M)$ more generally?

In the next section is a description of the connected components for an enveloping algebra, but it does not involve a separable, therefore finite dimensional, subalgebra.

3. **Representations of Lie algebras.** Enveloping algebras of Lie algebras provide concrete examples of infinite dimensional non-

commutative algebras and a good testing ground to try to describe the connected components of $\text{Mod}_A(M)$ for noncommutative algebras.

Let L be a finite dimensional Lie algebra and $U(L)$ its enveloping algebra. $U(L)$ is always infinite dimensional. Modules over L are exactly the same as modules over $U(L)$, so we define $\text{Mod}_L(M)$ to be $\text{Mod}_{U(L)}(M)$.

The cohomology groups of L are defined for each left L-module N (whereas the cohomology groups for an associative algebra are defined for two-sided modules) by

$$H^n(L, N) = \text{Ext}^n_{U(L)}(k, N)$$

where k is the trivial L-module, $L \cdot k = 0$. (See [10, p. 234].)

We are only interested in the first cohomology group for which we have the characterization

$$H^1(L, N) \cong \text{Der}\,(L, N)/\text{Inder}\,(L, N)$$

the derivations of L in N modulo the inner derivations.

Let $\rho: L \to \text{End}_k(M)$ be a representation of L. We make $\text{End}_k(M)$ into a left L-module by defining $au = [\rho(a), u]$. $\text{End}_k(M)$ is also a two-sided $U(L)$-module using the associated algebra homomorphism $\hat{\rho}: U(L) \to \text{End}_k(M)$. We also have the isomorphism $H^1(L, \text{End}_k(M), \rho) \cong H^1(U(L), \text{End}_k(M), \hat{\rho})$. For each derivation $\nu: L \to \text{End}_k(M)$ we define the cocycle $\hat{\nu}: U(L) \to \text{End}_k(M)$, extending ν to products of elements from L such as $\nu(ab) = \rho(a)\nu(b) + \nu(a)\rho(b)$. Clearly, inner derivations become cocycles. In the other direction, a cocycle for $U(L)$, when restricted to L, is a derivation.

Therefore, just as in the case of modules over an associative algebra, we conclude that the orbit of a representation ρ is open if and only if $H^1(L, \text{End}_k(M), \rho) = 0$, and the orbit is closed if and only if the representation is semisimple.

Now let k be an algebraically closed field of characteristic 0. The connected components of $\text{Mod}_L(M)$ are described by this theorem:

THEOREM 3.1. *The connected components correspond to the isomorphism classes of $L/\text{rad}\,L$-modules on M.*

Proof. The radical $\text{rad}\,L$ is the largest solvable ideal of L. The quotient is semisimple and by Levi's Theorem there is a subalgebra S isomorphic to $L/\text{rad}\,L$. The injection $i: L/\text{rad}\,L \to L$ mapping onto S gives a scheme morphism $i^*: \text{Mod}_L(M) \to \text{Mod}_{L/\text{rad}\,L}(M)$. If ρ and σ are in the same connected component of $\text{Mod}_L(M)$ then they are in the same component of $\text{Mod}_{L/\text{rad}\,L}(M)$. Since $L/\text{rad}\,L$ is semisimple its first cohomology groups are zero (First Whitehead

Lemma), and so all the orbits are open. Therefore all the orbits are closed and the orbits are the connected components. Thus $i^*(\rho)$ is isomorphic to $i^*(\sigma)$.

Now we will show that any representation ρ can be deformed to one in which rad L acts trivially. Let ρ be a representation of L. Consider the representation of rad L. Since rad L is solvable there exists an eigenvector $x \in M$ for the action of rad L. Let X be the line generated by x and let U be a complement to X. Also, let $R = $ rad L and $S = L/\text{rad } L$ viewed as a subalgebra of L. For each $t \in k$, $t \neq 0$, define the linear automorphism of M

$$\phi_t \colon X \oplus U \longrightarrow X \oplus U \colon (x, u) \longmapsto (tx, u) \ .$$

Let $\rho_t = \phi_t \cdot \rho$. Then for $r \in R$

$$\rho_t(r) \colon x \longmapsto \rho(r)(x)$$
$$u \longmapsto t\pi_X(\rho(r)(u)) + \pi_U(\rho(r)(u)) \ .$$

Now let $t = 0$. We get a representation for which R acts on M as on the direct sum $X \oplus M/X$, and for S the representation is isomorphic to $\rho|S$ since S-orbits are closed. We continue in this way until we have a representation of L with R acting on the sum of one-dimensional modules $X_1 \oplus \cdots \oplus X_\mu$, while the action of S is isomorphic to the original ρ. Call this new representation σ.

Now construct a deformation of σ defined by

$$\sigma_t(s + r) = \sigma(s) + t\sigma(r) \ .$$

This is actually a representation for each $t \in k$, since

$$\sigma_t([r_1, r_2]) = 0 = [\sigma_t(r_1), \sigma_t(r_2)]$$

for all r_1 and r_2 in R, because the action of R is diagonalized. Let $t = 0$ and the result is a representation for which R acts trivially.

Therefore in each connected component of $\text{Mod}_L(M)$ is a unique closed orbit of modules with rad L acting trivially. Notice, however, that this orbit is *not* in the closure of the other orbits.

In order to classify the connected components of $\text{Mod}_A(M)$ for infinite dimensional, noncommutative algebras, it would be useful to have such a characterization for the enveloping algebras in intrinsic terms, without recourse to the structure of the Lie algebras underlying them.

4. **Rational points.** The basic fact underlying this section is the following:

PROPOSITION 4.1. *Let ρ be an A-module structure on M. The*

rational points of the orbit X_ρ are the A-module structures isomorphic to ρ.

Proof. Let σ be a rational point of X_ρ. Then there is some $\phi \in \mathrm{Aut}_k(M)(\bar{k})$ such that $\phi \cdot \rho = \sigma$, where $\bar{k}$ is an algebraic closure of k. Now we can assume that ϕ is actually in $\mathrm{Aut}_A(M)(L)$ for some finite extension L. For example, let L be the field generated by the entries of a matrix representation of ϕ. Therefore, the extensions ρ_L and σ_L are isomorphic. But $\rho_L \cong \rho \oplus \cdots \oplus \rho$ and $\sigma_L \cong \sigma \oplus \cdots \oplus \sigma$, each sum with s factors where $s = \dim_k L$. By the Krull-Schmidt Theorem one sees that $\rho \cong \sigma$ as A-modules. □

Typically, when an algebraic group acts on a k-scheme the rational points in the orbit of a rational point include many points that are not equivalent by a rational point of the group. We make use of 4.1 to describe the connected components of the space $\mathrm{Mod}_A(M)(k)$ whose topology is the induced Zariski topology.

THEOREM 4.2. *Let k be an infinite field. Suppose A is finite dimensional with a decomposition as a semi-direct sum $S \oplus N$ of a semisimple subalgebra S and the radical N. Then two module structures ρ and σ are in the same connected component of $\mathrm{Mod}_A(M)(k)$ if and only if they have isomorphic Jordan-Hölder factors.*

Proof. The algebra morphism $S \to A$ induces a map $\mathrm{Mod}_A(M)(k) \to \mathrm{Mod}_S(M)(k)$. If two module structures ρ and σ are in the same component of $\mathrm{Mod}_A(M)(k)$ then they are in the same component of $\mathrm{Mod}_S(M)(k)$. Because S is semisimple all orbits of $\mathrm{Mod}_S(M)$ are open and so $\mathrm{Mod}_S(M)(k)$ is the disjoint union of a finite number of open sets. Therefore each one is closed as well. Each of these closed-open sets is the set of rational points of an orbit and so ρ and σ have the same Jordan-Hölder factors.

For the converse we deform ρ and σ to semisimple modules ρ_s and σ_s which are isomorphic. Since k is infinite, these deformations are given by morphisms from the connected set $(\mathrm{Spec}\, k[t])(k) = k$ into $\mathrm{Mod}_A(M)(k)$. Therefore ρ_s is in the same component as ρ and σ_s is in the same component as σ. Now ρ_s and σ_s are in the same component because they are in the image of an orbit map with domain $\mathrm{Aut}_k(M)(k)$, which is also connected since k is infinite.

PROPOSITION 4.3. *Let A be finite dimensional. Then the connected components of $\mathrm{Mod}_A(M)$ correspond to the Galois equivalence classes of semisimple $A \otimes \bar{k}$-modules on $M \otimes \bar{k}$.*

Proof. The components of $\mathrm{Mod}_A(M) \times \mathrm{Spec}\,\bar{k}$ correspond to the semisimple $A \otimes \bar{k}$-modules. The projection $\pi \colon \mathrm{Mod}_A(M) \times \mathrm{Spec}\,\bar{k} \to \mathrm{Mod}_A(M)$ maps connected components onto connected components. Since the fibers of the projection are the orbits of the Galois group, two connected components will be mapped to the same component if they have any Galois equivalent points. $\qquad\square$

Let ρ be a semisimple A-module structure isomorphic to $L_1^{m_1} \oplus \cdots \oplus L_s^{m_s}$, each L_i is a simple A-module. Then $\dim X = \dim \mathrm{Aut}_k(M) - \dim \mathrm{Aut}_A(\rho)$, since $\mathrm{Aut}_A(\rho)$ is the stabilizer of ρ. We can compute this using the dimensions of the Lie algebras, since $\mathrm{Aut}_A(\rho)$ is smooth by virtue of being an open subscheme of the affine space $\mathrm{End}_A(\rho)$. Therefore $\dim X = n^2 - \dim_k \mathrm{End}_A(\rho) = n^2 - \Sigma\, c_i m_i^2$, where c_i is the dimension of $\mathrm{End}_A(L_i)$.

PROPOSITION 4.4. *If A is semisimple and $\dim_k A = \dim_k M = n$, then the orbit of the left regular representation has the maximal dimension among the orbits of $\mathrm{Mod}_A(M)$, and the dimension is $n^2 - n$.*

Proof. Let $e_i = \dim_k L_i$, $c_i = \dim_k \mathrm{End}_A(L_i)$, and let u_i be the multiplicity of L_i in the left regular representation for the minimal left ideal L_i. Therefore $u_i = e_i/c_i$ since u_i is the dimension of L_i over $\mathrm{End}_A(L_i)$.

In order to maximize the orbit dimension, we need to minimize $\Sigma\, m_i^2 c_i$ over $(m_1, \cdots, m_s) \in N^s$, with the constraint that $\Sigma\, m_i e_i = n$. To do this we can use Lagrange multipliers and actually minimize over R^s.

Define $f, g \colon R^s \to R$ by

$$f(x_1, \cdots, x_s) = \Sigma\, x_i^2 c_i$$
$$g(x_1, \cdots, x_s) = \Sigma\, x_i e_i - n \ .$$

Now minimize f on the hyperplane given by $g(x) = 0$. We have

$$Df(x_1, \cdots, x_s) = (2c_1 x_1, \cdots, 2c_s x_s)$$
$$Dg(x_1, \cdots, x_s) = (e_1, \cdots, e_s) \ .$$

Solving $\lambda Df(x) = Dg(x)$ for $\lambda \in R$ and $x \in R^s$, gives $2\lambda c_i x_i = e_i$. Therefore $x_i = e_i/2\lambda c_i$. Substituting these values into g gives $\lambda = 1/2$, from which it follows that $x_i = e_i/c_i = u_i$.

The point $u = (u_1, \cdots, u_s)$ is actually a minimum since $D^2 f(u)$ is positive definite and so it is positive definite when restricted to the tangent space of $g^{-1}(0)$ at u. The dimension of this orbit is $n^2 - \Sigma\, u_i^2 c_i = n^2 - \Sigma\, u_i e_i = n^2 - n$.

5. An example. The structure of $\text{Mod}_A(M)$ has only been described for a few algebras. One would like to know the irreducible components, the open orbits, and the ordering of orbits given by $X_\sigma \leqq X_\rho$ if and only if X_σ is contained in the closure of X_ρ.

The details for $A = k[x]/(x^n)$ have been worked out by Gabriel [6] in the language of schemes and by Flanigan and Donald [4] in the language of deformation theory.

We will examine the structure of $\text{Mod}_A(M)$ for the three dimensional algebra $A = k[x, y]/(x, y)^2$. There is only one connected component because there is only one semisimple module in each dimension. In order to determine the open orbits we need to list the indecomposable A-modules [9, Prop. 5]. In A let $u_1 = \bar{x}$ and $u_2 = \bar{y}$. The radical of A is the ideal U generated by u_1 and u_2. Let M be an A-module, so UM is a submodule of M. Pick a complementary subspace V and write $M = V \oplus UM$. The scalar multiplication of u_1 and u_2 take the form

$$\begin{bmatrix} 0 & 0 \\ T_i & 0 \end{bmatrix}$$

with respect to the decomposition $V \oplus UM$. Any A-module is equivalent to a pair of linear maps in $\text{Hom}_k(V, W)$. The module is $V \oplus W$ as a vector space and multiplication by u_i has the form of the matrix above with $W = UM$.

Two modules given by the pairs (T_1, T_2) and (S_1, S_2) are isomorphic if there are automorphisms $P \in \text{Aut}_k(V)$ and $Q \in \text{Aut}_k(W)$ such that $S_i = QT_iP$ for $i = 1, 2$.

The indecomposable A-modules fall into three series: E_n, $n \geqq 0$; F_n, $n \geqq 0$; $G_n(\pi)$, $n \geqq 1$ and $\pi = \infty$ or a monic irreducible polynomial in $k[x]$. These modules are all distinct except that $E_0 = F_0$, the one-dimensional module isomorphic to A/U. The dimensions of the indecomposable modules are

$$\dim E_n = \dim F_n = 2n + 1$$
$$\dim G_n(\pi) = 2n(\deg(\pi)), \ \deg(\infty) = 1 \ .$$

The free module A is isomorphic to F_1. The modules E_n and F_n are dual.

On E_n the action of u_1 and u_2 are given by matrices T_1 and T_2 with $V = k^{n+1}$ and $W = k^n$.

$$T_1 = \begin{bmatrix} I_n & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \end{bmatrix} \qquad T_2 = \begin{bmatrix} \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} & I_n \end{bmatrix}$$

On F_n we have the transposed matrices:

$$T_1 = \begin{bmatrix} I_n \\ 0 \cdots 0 \end{bmatrix} \qquad T_2 = \begin{bmatrix} 0 \cdots 0 \\ I_n \end{bmatrix}.$$

For the module $G_n(\pi)$ with $\pi(x) = x^m - a_{m-1}x^{m-1} - \cdots - a_0$ the matrices T_1 and T_2 are $nm \times nm$ square matrices with the following form

$$T_1 = I_{nm} \qquad\qquad T_2 = \begin{bmatrix} B & 0 & 0 & \cdots & \cdot \\ N & B & 0 & \cdots & \cdot \\ 0 & N & B & \cdots & \cdot \\ \cdot & \cdot & & & \cdot \\ 0 & 0 & & \cdot & N\, B \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 & \cdots & 0 & a_0 \\ 1 & 0 & \cdots & 0 & a_1 \\ 0 & 1 & \cdots & 0 & a_2 \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ 0 & 0 & \cdots & 1 & a_{m-1} \end{bmatrix} \qquad N = \begin{bmatrix} 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Note that B is the companion matrix of π. Now for $\pi = \infty$ we have

$$T_1 = \begin{bmatrix} 0 & \cdots & 0 \\ I_{n-1} & & \cdot \\ & & \cdot \\ & & 0 \end{bmatrix} \qquad\qquad T_2 = I_n\,.$$

With this knowledge of the indecomposable A-modules we can state the theorem describing those modules whose orbits are open.

THEOREM 5.1.   *Let M be a finite dimensional A-module of dimension m. **The** orbit of M is open in $\mathrm{Mod}_A(k^m)$ if and only if M is isomorphic to $E_n^r \oplus E_{n+1}^s$ or to $F_n^r \oplus F_{n+1}^s$, where $n \geq 1$ and $m = r(2n+1) + s(2n+3)$.*

Proof. M is isomorphic to the direct sum of indecomposable modules $I_1 \oplus \cdots \oplus I_\nu$. Since Ext is additive in both factors

$$\mathrm{Ext}_A^1(M, M) = \bigoplus_{\lambda, \mu} \mathrm{Ext}_A^1(I_\lambda, I_\mu)\,.$$

So one needs to determine the Ext-groups for the indecomposable A-modules. The dimension of $\mathrm{Ext}_A^1(I, J)$ can be computed for indecomposable modules I and J. The computations may be found in [11]. We summarize with a table of the dimensions for $\mathrm{Ext}_A^1(I, J)$ and we use $d(\pi)$ for the degree of π.

J \ I	E_m	F_m	$G_m(\pi)$
E_n	$m + 2(n = 0)$ $0 \quad (1 \leq n \leq m + 1)$ $n - m - 1(m + 1 < n)$	0	0
F_n	$m + n + 2$	$n + 2(m = 0)$ $0 \quad (1 \leq m \leq n + 1)$ $m - n - 1(n + 1\ m)$	$md(\pi)$ $\min(m,\ n)d(\pi)(\pi = \phi)$
$G_n(\phi)$	$nd(\phi)$	0	$0 \quad (\pi \neq \phi)$

From the table one sees that in order for $\mathrm{Ext}\,(I \oplus J,\ I \oplus J) = \mathrm{Ext}\,(I, I) \oplus \mathrm{Ext}\,(J, I) \oplus \mathrm{Ext}\,(J, I) \oplus \mathrm{Ext}\,(J, J)$ to be zero, I and J must be E_n and E_{n+1} or they must be F_n and F_{n+1}. The theorem then follows directly with repeated use of this observation. $\square$

In $\mathrm{Mod}_A(k^m)$ for m odd, the irreducible components each have an open orbit and so the closure of that open orbit is the irreducible component. However, for the even dimensional modules there is an additional irreducible component not given by an open orbit. It is the one containing the orbits of the modules of the form

$$G_{i_1}(\pi_1) \oplus \cdots \oplus G_{i_r}(\pi_r)$$

where $\Sigma i_j \deg (\pi_j) = m$. In general — when the polynomials $\pi_1, \cdots, \pi_r$ are all distinct — the codimension of the orbit is m.

In [5] Flanigan and Donald give a formula for the number of irreducible components in $\mathrm{Mod}_A(k^m)$. The number is

$$1 \qquad \text{for } m = 1$$
$$2[(m + 3)/6] \quad \text{for } m > 1 \text{ and odd}$$
$$2[m/6] + 1 \quad \text{for } m \text{ even}$$

where the square brackets denote the greatest integer function.

6. Deformation theory of modules. The theory of deformations of modules follows in the spirit of Gerstenhaber's work on deformations of algebras in [8]. Viewed in a geometric setting deformation theory concentrates on the $k[[t]]$-valued points of the scheme $\mathrm{Mod}_A(M)$. These points are the deformations, and we will call them formal deformations. (In [4] the term used is "generic deformation.") Strictly speaking the functor $\mathrm{Mod}_A(M)$ need not even

be representable; for example, M could be infinite dimensional, and the ideas of deformation theory could be applied while the algebraic geometry could not. However, no work has been done in situations where $\mathrm{Mod}_A(M)$ is not a scheme, even though that fact is not used.

Suppose we are given an actual curve or one-parameter family of module structures, that is, a scheme morphism $c\colon \mathrm{Spec}\, k[t] \to \mathrm{Mod}_A(M)$. Taking the Taylor expansion of c at 0 gives a formal deformation of the module structure $\rho = c(0)$. Functorially, we are considering the localization morphism $\alpha\colon k[t] \to k[\![t]\!]$ inducing $\mathrm{Spec}\,\alpha\colon \mathrm{Spec}\, k[\![t]\!] \to \mathrm{Spec}\, k[t]$ and the composition $c\circ\mathrm{Spec}\,\alpha$, which gives a $k[\![t]\!]$-valued point of $\mathrm{Mod}_A(M)$. Then we define the set of formal deformations of ρ to be the scheme morphisms $\beta\colon \mathrm{Spec}\, k[\![t]\!] \to \mathrm{Mod}_A(M)$ such that $\beta(x) = \rho$ where x is the unique closed point of $\mathrm{Spec}\, k[\![t]\!]$. Algebraically β is simply an $A \otimes k[\![t]\!]$-module structure on $M \otimes k[\![t]\!]$ which restricts to ρ when $t = 0$. Let $\pi\colon k[\![t]\!] \to k\colon t \mapsto 0$ be the projection. Then the formal deformations of ρ are $\mathrm{Mod}_A(M)(\pi)^{-1}(\rho)$. Not every formal deformation need be the Taylor expansion of an actual deformation; over the complex field there are convergence problems.

It is customary (see [4]) to define a formal deformation of ρ to be an $A \otimes K$-module structure on $M \otimes K$, $K = k((t))$, of the form

$$\rho_t(a)(m) = \rho(a)(m) + tR_1(a)(m) = t^2R_2(a)(m) + \cdots$$

for $a \in A$, $m \in M$, $R_i\colon A \to \mathrm{End}_k(M)$. Then it is extended to $A \otimes K$ and $M \otimes K$ to make it K-linear in each factor. The advantage in this definition is to allow everything to be done over a new base field K. However, it is not geometric because there is no projection $K \to k$, so there is no way to recover ρ from an arbitrary $A \otimes K$-module structure on $M \otimes K$. The two different definitions are equivalent: from a $k[\![t]\!]$-point we get a module structure on $M \otimes K$ simply by extending the scalars. In the other direction we need to know that ρ_t is of the special form above, which says that ρ_t is actually defined over $k[\![t]\!]$.

In a similar fashion one can define the formal deformations of any k-valued point x in a k-scheme X as the set of $k[\![t]\!]$-points lying over x. For example, the set of formal deformations of the identity I in $\mathrm{Aut}_k(M)$ will be of interest to us. This set forms a subgroup of $\mathrm{Aut}_k(M)(k[\![t]\!])$ and there is an action of the group of deformations of I on the set of deformations of ρ. Explicitly, a deformation of I can be written in the form $I + tu_1 + t^2u_2 + \cdots$ where each u_i is in $\mathrm{End}_k(M)$. As with deformations of ρ we may define a formal deformation of I to be a K-linear automorphism of $M \otimes K$ having the form above.

One says that two formal deformations ρ_t and σ_t are *equivalent* if they are in the same orbit for the action of the group of formal deformations of I. A module ρ is *rigid* if every formal deformation of ρ is equivalent to the zero-deformation (the canonical extension of ρ to $M \otimes k[\![t]\!]$). That means that all formal deformations of ρ are equivalent, so there is just one orbit in the set of formal deformations of ρ.

Immediately one sees that if the orbit of ρ is an open subscheme of $\mathrm{Mod}_A(M)$, then ρ is rigid. At ρ the $k[\![t]\!]$-points of X and the $k[\![t]\!]$-points of $\mathrm{Mod}_A(M)$ are the same, $k[\![t]\!]$ being a local algebra.

If ρ is a nonsingular point of $\mathrm{Mod}_A(M)$ then the converse is also true. Let ν be a tangent vector at ρ. Then ν is tangent to an actual deformation, which is a curve through ρ, and therefore there is a formal deformation $\rho_t = \rho + t\nu + \cdots$. But ρ_t is equivalent to the zero-deformation by an automorphism $I + tu_1 + \cdots$. An easy calculation shows that the cocycle ν is the coboundary of u_1, i.e., $\nu(a) = [\rho(a), u_1]$. Therefore $H^1(A, \mathrm{End}_k(M), \rho) = 0$ and it follows that X_ρ is open.

If ρ is a singular point it may be that ρ is not reduced, so consider the reduced subscheme. If ρ is nonsingular in the reduced scheme and if ρ is a rigid module, then the orbit X_ρ is open in $\mathrm{Mod}_A(M)_{\mathrm{red}}$ by a similar argument.

Over an algebraically closed base field k rigidity of the module structure ρ is equivalent to the orbit X_ρ being open. If k is not algebraically closed one should consider the R-points for all complete discrete valuation rings R.

PROPOSITION 6.1. *Let k be algebraically closed. The module structure ρ is rigid if and only if X_ρ is open in the reduced subscheme $\mathrm{Mod}_A(M)_{\mathrm{red}}$.*

Proof. Let $X = X_\rho$ and $Y = \mathrm{Mod}_A(M)_{\mathrm{red}}$ and $x = \rho$. We have the situation in which $X \subset Y$ is a subvariety and $x \in X$ is a point at which the $k[\![t]\!]$-points of X and Y are the same. If X is not open at x, then X is a proper subvariety in a neighborhood of x. Let Z be any subvariety of Y such that x is isolated in $X \cap Z$. Then there is a nontrivial morphism $\alpha \colon \mathrm{Spec}\, k[\![t]\!] \to Z$ such that $\alpha(p) = x$ where p is the closed point of $\mathrm{Spec}\, k[\![t]\!]$. Such an α is a $k[\![t]\!]$-point of Y based at x which is not a $k[\![t]\!]$-point of X. $\square$

The following is an example of a module structure which is rigid and whose orbit is open in the reduced scheme but not in $\mathrm{Mod}_A(M)$. Let A be $k[Z_p]$ where k is a field of characteristic p, and let ρ be the trivial one-dimensional representation. Then $\mathrm{Mod}_A(M) \approx$

Spec $k[x]/(x^p - 1)$ which has one closed point, the representation ρ,
but is not reduced. The orbit of ρ is the closed point, the sub-
scheme corresponding to Spec $k[x]/(x - 1)$, but it is not open because
it does not fill up the tangent directions. In this case the orbit of
ρ and the reduced scheme are the same.

Several of the results proved in [4] about deformations of
modules can be proved in the context of the scheme of module
structures. We will give several examples in which a property of
modules is stable under formal deformations because it is actually
an open condition, a property which is true on an open subscheme
of $\mathrm{Mod}_A(M)$. Such open properties are cyclic, faithful, irreducible,
projective, and injective.

First, consider a cyclic A-module structure ρ with generator
$x \in M$. The evaluation map $ev_x: A \to M: a \mapsto \rho(a)(x)$ is surjective. One
would expect that x is a generator for any module structure in
some open neighborhood of ρ. Define the scheme morphism

$$ev: \mathrm{Mod}_A(M) \times M \longrightarrow \mathrm{Hom}_k(A, M)$$

where $ev(\rho, x)$ is the morphism that sends a to $\rho(a)(x)$. The functor
$\mathrm{Hom}_k(A, M)$ is represented by the affine k-scheme Spec $S(A^* \otimes M)$
where S denotes the symmetric algebra. This scheme is of finite
type if and only if A is finite dimensional. Now in $\mathrm{Hom}_k(A, M)$ is
the open subscheme of surjective maps $\mathrm{Sur}_k(A, M)$ whose R-points
are the surjective R-module homomorphisms from $A \otimes R$ to $M \otimes R$.
Thus $ev^{-1}(\mathrm{Sur}_k(A, M))$ is an open subscheme of $\mathrm{Mod}_A(M) \times M$ con-
sisting of pairs (ρ, x) for which x is a generator for ρ. Also, we
may fix the generator x. Then $ev_x^{-1}(\mathrm{Sur}_k(A, M))$ is open in the
scheme of module structures and contains ρ; this set consists of
the modules that are cyclic with x as a generator.

Not surprisingly, for formal deformations ρ_t of ρ, the element
$x \otimes 1$ is a generator for ρ_t. This is obvious from the algebraic
geometry.

Recall that a module is faithful if its representation map ρ:
$A \to \mathrm{End}_k(M)$ is injective. Since M is finite dimensional, A must
also have finite dimension. In order to show that the faithful
modules form an open subscheme of $\mathrm{Mod}_A(M)$ consider the open
subscheme $\mathrm{SInj}_k(A, M)$ contained in $\mathrm{Hom}_k(A, M)$ whose R-points are
the split injective maps $\phi: A \otimes R \to M \otimes R$. Clearly for $R = k$ we
get the injective maps from A to M. To see that $\mathrm{SInj}_k(A, M)$ is
really a scheme, we identify it with $\mathrm{Sur}_k(M^*, A^*)$ where M^* and
A^* are the k-duals of M and A. A morphism $\phi: A \otimes R \to M \otimes R$
is split injective if and only if $^t\phi: (M \otimes R)^* \to (A \otimes R)^*$ is surjec-
tive. If we only required that the R-points be injective maps then

the functor would not be representable.

Now the subscheme $\mathrm{SInj}_k(A, M) \cap \mathrm{Mod}_A(M)$ is open in $\mathrm{Mod}_A(M)$ and the k-points are the faithful A-module structures on M. The R-points are the split faithful $A \otimes R$-module structures on $M \otimes R$. It is immediate that any formal deformation of a faithful module is faithful.

We will briefly consider the classes of simple modules, projective modules, and injective modules. Let k be algebraically closed and let $\rho\colon A \to \mathrm{End}_k(M)$ be a simple A-module structure. Since every closed point in the connected component of ρ has the same Jordan-Hölder factors as ρ, we see that the orbit of ρ is actually the connected component containing ρ in the reduced subscheme $\mathrm{Mod}_A(M)_{\mathrm{red}}$. We also see that ρ is rigid.

It is unclear how to use this argument to show that simple modules are rigid when k is not algebraically closed. There is, however, an algebraic proof in [4].

Now since $\mathrm{Ext}_A(E, F) = 0$ whenever E is injective or F is projective, the orbits of injective modules and the orbits of projective modules are open, and therefore injectives and projectives are rigid.

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