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Symplectic Flows on the Open Ball

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We construct a complete, infinitesimally symplectic vectorfield on the open unit ball in $\mathbf{R}^4$ with no zeros. This shows that there are symplectic diffeomorphisms of the ball with no fixed points.

INTRODUCTION

In an earlier paper [4] we raised the question: Does every symplectic diffeomorphism of the open unit ball, B^{2n}, in $\mathbf{R}^{2n}$ have a fixed point? We showed that there is an open neighborhood of the identity map in the C^1 fine topology for which the answer is yes. In this paper it is shown that the answer is no in general. There is a complete, infinitesimally symplectic vectorfield on B^4 without any zeros. Thus there are symplectic diffeomorphisms in any C^1 weak neighborhood of the identity with no fixed points. This shows that the two-dimensional results are special, for in the open disk every symplectic diffeomorphism has a fixed point, a result Bourgin proved in greater generality for orientation preserving homeomorphisms that preserve a finitely additive measure that is positive on open sets [3]. Furthermore, a complete, infinitesimally symplectic vectorfield on B^2 has a zero, as we show in Section 2.

The question of volume preserving, orientation preserving diffeomorphisms on the open ball was resolved in 1976 by Asimov [2] with his construction of a divergent free vectorfield on B^3, which is complete and has no zeros. This easily extends to all dimensions. In particular, on B^4 one had a volume and orientation preserving diffeomorphism with no fixed point, but not a symplectic diffeomorphism.

1. THE EXAMPLE

In this section we construct a complete, infinitesimally symplectic (Hamiltonian) vectorfield on B^4, the open unit ball in $\mathbf{R}^4$. A vectorfield will

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be complete on B^4 if it extends to a vectorfield on the closure $\bar{B}^4$ and is tangent to the boundary. So we look for a real valued function f on $\bar{B}^4$ such that the Hamiltonian vectorfield

$$X_f = (\partial f/\partial y_1, \partial f/\partial y_2, -\partial f/\partial x_1, -\partial f/\partial x_2)$$

is tangent to the unit sphere. The coordinates on $\mathbf{R}^4$ are (x_1, x_2, y_1, y_2) and the symplectic form is $\omega = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$.

It is also convenient to use complex coordinates in $\mathbf{R}^4 = \mathbf{C}^2$, $z_j = x_j + iy_j$, $j = 1, 2$. Then $X_f = -i(\operatorname{grad} f)$ and X_f is tangent to the unit sphere when $-i(\operatorname{grad} f) \cdot z = 0$, for each z on the unit sphere. Here the dot means the real inner product, which is the real part of the Hermitian inner product. This condition is equivalent to $(\operatorname{grad} f) \cdot iz = 0$. Thus, $\operatorname{grad} f$ is perpendicular to the linear vectorfield $J(z) = iz$ when restricted to the unit sphere. The integral curves of J on the unit sphere are the circles of the Hopf fibration, and so f is constant on those circles.

The Hopf fibration is the projection of S^3 onto S^2, where S^2 is identified with $\mathbf{CP}^1$, given by

$$\pi : S^3 \to \mathbf{CP}^1 : (z_1, z_2) \to (z_1 : z_2).$$

The point z on S^3 is mapped to the complex line in $\mathbf{C}^2$ containing z. Now we collapse $\bar{B}^4$ by identifying boundary points if they lie in the same fiber of π, while leaving the interior points alone. The quotient space is $\mathbf{CP}^2$ and the quotient map may be given explicitly

$$h : \bar{B}^4 \to \mathbf{CP}^2 : (z_1, z_2) \to (1 - |z|^2 : z_1 : z_2).$$

One may check that for $|z| < 1$, $h(z) = h(w)$ implies that $z = w$ or $|w| < 1$. This shows that h is one–to–one on the interior. On the boundary S^3, $h(z) = (0 : z_1 : z_2)$ and so h is the Hopf fibration onto the $\mathbf{CP}^1$ given by $z_0 = 0$.

To see that h is a diffeomorphism from the open ball to the complement of $z_0 = 0$, we can show that the closed disks through the origin whose boundaries are the circles in the fibration are each mapped onto a complex projective line through $(1 : 0 : 0)$. Let (z_1, z_2) be a point on S^3. The unit disk spanned by its fiber and $(0, 0)$ is mapped onto the $\mathbf{CP}^1$ containing $(1 : 0 : 0)$ and $h(z_1, z_2) = (0 : z_1 : z_2)$. The boundary circle is mapped to the single point $(0 : z_1 : z_2)$.

The function $\phi : \mathbf{CP}^2 \to \mathbf{R} : (z_0, z_1, z_2) \to |z_0|^2 + 2|z_1|^2 + 3|z_2|^2$ is a Morse function with three critical points $p_0 = (1 : 0 : 0)$, $p_1 = (0 : 1 : 0)$, $p_2 = (0 : 0 : 1)$. From ϕ we construct a function whose critical points all lie in $h(S^3)$. Pick a point q in $h(S^3)$ distinct from p_1 and p_2, say, $q = (0 : 1 : 1)$. Construct an isotopy of diffeomorphisms $F_t : \mathbf{CP}^2 \to \mathbf{CP}^2$ such that $F_0 = I$, while F_1 fixes p_1 and p_2 and $F_1(p_0) = q$. The family F_t can be constructed by connecting p_0 with q by a narrow tube. Let F_t be the identity outside the tube and let F_t push p_0 toward q inside the tube.

The function $\phi \circ F_1^{-1}$ has three critical points p_1, p_1, and q, all lying in $h(S^3)$. Hence the function $f = \phi \circ F_1^{-1} \circ h$ has no critical points in the interior of the ball $\bar{B}^4$, and it is constant on the fibers of the Hopf fibration. The vectorfield X_f has no zeros in B^4 and is complete. Thus the flow Φ_t of X_f consists of symplectic diffeomorphisms with no fixed points for values of t close enough to 0.

In higher dimensions the same construction, mapping $\bar{B}^{2n}$ onto $\mathbf{CP}^n$ with the boundary S^{2n-1} fibering over a complex hyperplane $\mathbf{CP}^{n-1}$, enables us to give complete, Hamiltonian vectorfields on B^{2n} with no zeros.

Remark. In an earlier article [4] we showed that there is a neighborhood of the identity in the C^1 fine topology on the space of symplectic diffeomorphisms $\text{Diff}(B^{2n}, \omega)$ such that any map in this neighborhood has a fixed point. However, the example constructed above shows that there is no neighborhood of the identity in the C^1 weak topology with the same property, since the one parameter group $\Phi : \mathbf{R} \to \text{Diff}(B^4, \omega) : t \to \Phi_t$ is continuous in the weak topology and Φ_t has no fixed points for t small enough.

2. Two Dimensions

In this section we show that a complete, infinitesimally symplectic vectorfield on the open disk has at least one zero. Again the results are special in the two dimensional setting. See the article by Simon [5] and Appendix 9 in Arnold's book [1].

Let $X = X_f$ be an infinitesimally symplectic vectorfield on the open unit disk $D \subset \mathbf{R}^2$. In polar coordinates $X_f(r, \theta) = (1/r)(\partial f/\partial \theta, -\partial f/\partial r)$ so that Hamilton's equations are

$$\dot{r} = (1/r)\, \partial f/\partial \theta,$$
$$\dot{\theta} = (-1/r)\, \partial f/\partial r.$$

Assume X_f is complete. Thus, as one approaches the boundary of the disk the $\dot{r}$ component of the vectorfield must go to zero. Thus, $\partial f/\partial \theta$ goes to zero at the boundary. From this it follows that f has the same limit at all boundary points, whether finite or infinite. Let $\bar{f} : \bar{D} \to \mathbf{R} \cup \{\infty\}$ be the continuous extension of f to the closed disk. Now $\bar{f}$ attains both its maximum and its minimum. If they are both on the boundary then f is constant and $X_f = 0$. Otherwise f has an extreme point in the interior D and X_f has a zero at that point. We conclude that X_f has at least one zero in D.

Questions. Let G be a connected, abelian Lie group (a product of $\mathbf{R}^k$ with a torus). Does every symplectic action of G on the disk D have a fixed point? Is it true for an open, dense set of actions? Such an action comes from a finite set of functions $f_1, \ldots, f_n$ in involution. One can show that if one of them has an isolated critical point, then that point is a fixed point of the action, but when none of the critical points is isolated the analysis is more complicated.

Are there compact Lie groups acting symplectically on D with no fixed points?

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