THE EXTENDED FISHER-HARTWIG CONJECTURE FOR SYMBOLS WITH MULTIPLE JUMP DISCONTINUITIES

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Dedicated to Harold Widom on his sixtieth birthday

The asymptotic expansions of Toeplitz determinants of certain symbols with multiple jump discontinuities are shown to satisfy a revised version of the conjecture of Fisher and Hartwig. (This paper has appeared in *Operator Theory: Advances and Applications*, 71, 1994, 16-28.)

§1. Introduction

The Toeplitz matrix $T_n[\phi]$ is said to be generated by the function ϕ if

$$T_n[\phi] = (\phi_{i-j}),\ i,j = 0,\ldots,n-1.$$

where

$$\phi_n = \frac{1}{2\pi}\int_0^{2\pi} \phi(\theta)e^{-in\theta}d\theta$$

is the nth Fourier coefficient of ϕ. Define the determinant

$$D_n[\phi] = \det(T_n[\phi]),\ \ i,j = 0,\ldots,n-1. \tag{1}$$

The Fisher-Hartwig Conjecture [8] concerns the asymptotic behavior of the determinants of Toeplitz matrices for a certain class of singular symbols. These symbols are of the form

$$\phi(\theta) = b(\theta)\prod_{r=1}^{R} t_{\beta_r}(\theta-\theta_r)u_{\alpha_r}(\theta-\theta_r) \tag{2}$$

where

$$t_\beta(\theta) = \exp[-i\beta(\pi - \theta)], \quad 0 < \theta < 2\pi \tag{3}$$

$$u_\alpha(\theta) = (2 - 2\cos\theta)^\alpha, \quad \mathrm{Re}\,\alpha > -\frac{1}{2} \tag{4}$$

and $b : \mathbf{T} \to \mathbf{C}$ is a smooth non-vanishing function with zero index. Note that the function ϕ may have jump discontinuities, zeros, and/or singularities.

The first general results on the conjecture, which we will describe shortly, were obtained by Widom [9], who showed that it was true for $\mathrm{Re}\,\alpha_r > -1/2$ and $\beta_r = 0$ for all r. The conjecture was then extended by several authors to restricted values of the parameters β_r and α_r. In particular, it is true if $|\mathrm{Re}\,\beta_r| < 1/2$ and $|\mathrm{Re}\,\alpha_r| < 1/2$, and true in the case of one singularity for arbitrary β where $\alpha = 0$. A history of this work can be found in [7].

Recently, in an investigation of the distribution of eigenvalues of Toeplitz matrices it was shown that in some simple and unexpected cases the original conjecture was false [3, 2]. Earlier Böttcher and Silbermann [5] had also shown that the conjecture did not hold in the case of integer parameter values. A revised conjecture subsuming both kinds of counter-examples was formulated in [3].

The purpose of this paper is to prove the original conjecture in some additional cases, give an example underpinning the revised conjecture, and to discuss the implications for Toeplitz eigenvalues. We begin with a description of the new conjecture which takes into account the possibility of multiple representations of the symbol in the form specified by (2).

Conjecture *Suppose*

$$\phi(\theta) = b^i(\theta) \prod_{r=1}^{R} t_{\beta_r^i}(\theta - \theta_r) u_{\alpha_r^i}(\theta - \theta_r) \tag{5}$$

for values $\beta_1^i, \ldots, \beta_R^i, \alpha_1^i, \ldots, \alpha_R^i$ *and smooth nonzero functions* $b^i(\theta)$ *each with winding number zero for* $i = 1, 2, \ldots$. *(When* $R > 1$ *there is a countable number of different representations. Notice that* $|b^i|$ *is independent of* i.*) Define*

$$\begin{aligned}
\Omega(i) &= \sum_{r=1}^{R} \left((\alpha_r^i)^2 - (\beta_r^i)^2 \right) && (6)\\
\Omega &= \max_i \mathrm{Re}\,[\Omega(i)] && (7)\\
\mathcal{S} &= \{ i \mid \mathrm{Re}\,[\Omega(i)] = \Omega \}. && (8)
\end{aligned}$$

Then as $n \to \infty$,

$$D_n[\phi] = \sum_{i \in \mathcal{S}} G[b^i]^n n^{\Omega(i)} E[b^i, \alpha_r^i, \beta_r^i, \theta_r] + \mathrm{o}(G[|b|]^n n^{\Omega}) \tag{9}$$

where $G[b^i] = \exp(1/2\pi \int_0^{2\pi} \log b^i(\theta) d\theta)$ is the geometric mean of b^i, $G[|b|]$ is the geometric mean of any of the $|b^i|$, since they are all the same, and where $E[b^i, \alpha_r^i, \beta_r^i, \theta_r]$ is a constant described as follows. Suppressing the superscript i, we factor

$$b(\theta) = G[b] b_+(\exp(i\theta)) b_-(\exp(-i\theta))$$

where b_+ extends analytically inside the unit circle and b_- extends analytically outside the unit circle, and $b_+(0) = b_-(\infty) = 1$. Define

$$\begin{aligned} E[b, \alpha_r, \beta_r, \theta_r] &= E[b] \prod_{r=1}^{R} b_-(\exp(i\theta_r))^{-\alpha_r - \beta_r} b_+(\exp(-i\theta_r))^{-\alpha_r + \beta_r} \\ &\times \prod_{1 \le s \ne r \le R} [1 - \exp(i(\theta_s - \theta_r))]^{-(\alpha_r + \beta_r)(\alpha_s - \beta_s)} \\ &\times \prod_{r=1}^{R} G(1 + \alpha_r + \beta_r) G(1 + \alpha_r - \beta_r) / G(1 + 2\alpha_r), \end{aligned} \tag{10}$$

where G is the Barnes G-function, $E[b] = \exp(\sum_{k=1}^{\infty} k s_k s_{-k})$, and $s_k := [\log b(\theta)]_k$.

The original statement of Fisher-Hartwig was very similar except that the function ϕ was assumed to be of one fixed form (2). It is important to note that in the previous work cited only one representation yielded the maximum in Ω. In what follows we will show that (9) holds for a function ϕ with $\alpha = 0$ and $-1 < \operatorname{Re} \beta_r \le 0$ (or with $\alpha = 0$ and $0 \le \operatorname{Re} \beta_r < 1$). This is done in section 2. In section 3 we will show that (9) also holds in some examples and discuss the implications for eigenvalues. The results for section 2 agree with the original conjecture; however, the results of section 3 agree only with the revised conjecture.

§2. Localization theorem for $-1 < \operatorname{Re} \beta_r < 0$

We begin this section by showing that in the case $\alpha_r = 0$ and $-1 < \operatorname{Re} \beta_r \le 0$, there is only one representation in (5). It is thus not surprising that section 2 uses an adaptation of older techniques.

Consider a symbol of the form

$$\phi(\theta) = b(\theta) \prod_{r=1}^{R} t_{\beta_r}(\theta - \theta_r). \tag{11}$$

Other representations of ϕ correspond to changing β_r to $\beta'_r = \beta_r + j_r$, where $j_r \in \mathbf{Z}$ such that $\sum j_r = 0$. Now assume that $-1 < \operatorname{Re}\beta_r < 0$ and we will show that this representation is the only one that gives the maximum in (6). Let $\beta_r = x_r + iy_r$. To maximize (6) is equivalent to minimizing

$$f(j) := \sum_{r=1}^{R-1} (x_r + j_r)^2 + (x_R - \sum_{r=1}^{R-1} j_r)^2 \tag{12}$$

over all $j = (j_1, \ldots, j_{R-1}) \in \mathbf{Z}^{R-1}$. Expanding this out we obtain

$$f(j) = 2\sum_{r=1}^{R-1} j_r^2 + 2\sum_{r=1}^{R-1} (x_r - x_R) j_r + 2\sum_{l<r} j_l j_r. \tag{13}$$

Routine algebra rewrites this as $f(j) = j^T A j + 2b^T j$, where

$$A = \begin{bmatrix} 2 & 1 & 1 & \ldots & 1 \\ 1 & 2 & 1 & \ldots & 1 \\ & \vdots & & & \\ 1 & 1 & \ldots & 1 & 2 \end{bmatrix} \tag{14}$$

and $b = (x_1 - x_R, \ldots, x_r - x_R, \ldots, x_{R-1} - x_R)$.The matrix A is positive definite, so that we have a standard linear algebra problem of minimizing a function which is quadratic plus linear. The minimum over $\mathbf{R}^{R-1}$ is given by $-A^{-1}b$, but we need the minimum on the integral lattice points. Because of the convexity of the function being minimized, the minimal integer points occur at the those integer points surrounding the minimum $-A^{-1}b$. Computing A^{-1}, (either by noting that A is a circulant matrix, by Cramer's Rule, or by guessing), we find

$$A^{-1} = \frac{1}{R} \begin{bmatrix} R-1 & -1 & -1 & \ldots & -1 \\ -1 & R-1 & -1 & \ldots & -1 \\ & \vdots & & & \\ -1 & -1 & \ldots & -1 & R-1 \end{bmatrix} = I + \frac{1}{R} \begin{bmatrix} -1 & \ldots & -1 \\ & \vdots & \\ -1 & \ldots & -1 \end{bmatrix} \tag{15}$$

Let the minimum in $\mathbf{R}^{R-1}$ be the vector $z = -A^{-1}b$. Then

$$z_r = \frac{R-1}{R} x_r - \frac{1}{R}(x_1 + \ldots + \hat{x}_r + \ldots + x_R), \tag{16}$$

where the hat indicates x_r is omitted. From the assumption that $-1 < x_r < 0$ we see that $-1 < z_r < 1$, and that the minimal integer point j must have $j_r = 0, -1, 1$. The expression $f(j) = j^T Aj + 2b^T j$ is 0 for $j = 0$, and we will show that for any other choice of j the expression is positive. First, this is easy to check for $R = 2$. Next, if any $j_r = 0$, then the problem reduces to the case of $R - 1$ variables. This leaves us the situation in which all the j_r are 1 or -1. If all of them have the same sign, then it is also easy to see that $f(j)$ is positive. Therefore, let us assume that m of the j_r's are -1 and p of them are 1, with $m + p = R - 1$. The third term in $f(j)$, formula (13), is

$$2\sum_{l<r} j_l j_r = 2(\binom{p}{2} + \binom{m}{2} - pm) \tag{17}$$

Using the restrictions on the x_r we show that

$$f(j) > 2(m + \binom{p}{2} + \binom{m}{2} - pm) \tag{18}$$

Now the right side factors as $(p-m)(p-m-1)$, from which we see that it is not possible for positive integer values of p and m to make this expression negative. Thus, $f(j) > 0$ for all integer points j whose components are 0,1, or -1. This means that 0 is the unique minimum for (6) as j ranges over $\mathbf{Z}^{R-1}$.

We now restrict our attention to piecewise continuous symbols of the form

$$\phi(\theta) = b(\theta) \prod_{r=1}^{R} t_{\beta_r}(\theta - \theta_r) \tag{19}$$

where $b(\theta)$ is non-zero, sufficiently smooth, has winding number zero, and $-1 < \mathrm{Re}\,\beta_r < 0$. The Toeplitz operator $T[\phi]$ is represented by the matrix (ϕ_{i-j}), $i, j \geq 0$. The important properties of the Toeplitz operator $T[\phi]$ and the corresponding finite matrices are summarized in [7]. We list some of these below and state some simple consequences.

Let ℓ_p^μ, $1 < p < \infty$, be the weighted space of sequences $x = \{x_n\}$ satisfying $\sum_{n=1}^{\infty} |x_n|^p (n+1)^{p\mu} < \infty$. The class M_p^μ is the collection of all functions $a \in L^1$ such that the convolution $a * x \in \ell_p^\mu$ for all sequences x with finite support and

$$\|a\|_{M_p^\mu} := \sup\{\|a * x\|_p^\mu / \|x\|_p^\mu : x \neq 0\} < \infty. \tag{20}$$

It is clear that the Toeplitz operator $T[a] = (a_{i-j})$, $i, j \geq 0$, and the Hankel operator $H[a] = (a_{i+j})$, $i, j \geq 0$, are bounded on ℓ_p^μ if $a \in M_p^\mu$, and it also can be shown that M_p^μ is a Banach algebra under the above norm.

The closure of the trigonometric polynomials in M_p^μ is denoted by C_p^μ and is contained in the set of continuous functions in L^1. Likewise, $PC_{p,\mu}$ is the closure of the piecewise constant functions in M_p^μ. While the exact description of $PC_{p,\mu}$ is not known, it is true that if $-1/p < \mu < 1/q$, then a function of the form (19) is in $PC_{p,\mu}$ if b has finite total variation. Also, define $W_{r,s}^{\alpha,\beta}$ to be all functions in $L^1(\mathbf{T})$ satisfying $\{f_n, n \geq 0\} \in \ell_s^\beta$ and $\{f_n, n \leq 0\} \in \ell_r^\alpha$. It is known that $W_{1,1}^{\alpha,\beta} \cap W_{p,q}^{\gamma,\delta}$ is an algebra when $\gamma > 1/q$, $\delta > 1/p$, $\alpha \geq 0$, $\beta \geq 0$, and $1/p + 1/q = 1$. From now on it will be assumed that b does not vanish, has index zero, and is sufficiently smooth.

The next question to address is the invertibility of the Toeplitz operators $T[\phi]$ and the applicability of the finite section method to these operators. The answers are contained in the following theorems.

Theorem 1 *Let $p > 1$, $1/p + 1/q = 1$, $-1/p < \mu < 1/q$. Suppose*

(a) b does not vanish, has index zero and has finite total variation.

(b) $-1/p < \mu + \operatorname{Re}\beta_r < 1/q$ for all r.

Then $T[\phi]$ is invertible on ℓ_p^μ.

Proof. The operators $T[b]$, $T[t_{b_r}]$ are invertible on ℓ_p^μ due to Proposition 6.24 of [7], and thus, by Propositions 6.29 and 6.32 of [7], the operator $T[\phi]$ is Fredholm of index zero, and hence, invertible.

Theorem 2 *Let $p > 1$. Suppose b satisfies (a) of the hypothesis of Theorem 1 and, in addition, $-\operatorname{Re}\beta_r < 1/q$, $-\operatorname{Re}\beta_r - 1/p < \mu < 1/q$. Then $T_n[\phi]$ is invertible for all n sufficiently large and $T_n[\phi]^{-1}$ converges strongly to $T[\phi]^{-1}$ on ℓ_p^μ.*

Proof. This is immediate from Theorem 7.45 in [7].

Theorem 3 *If $p > 2$, $\epsilon > 0$, then the embeddings $\ell_p^\mu \to \ell_2^{\mu-1/2+1/p-\epsilon}$, $\ell_2^\mu \to \ell_q^{\mu-1/2+1/p-\epsilon}$, $\ell_2^\mu \to \ell_p^\mu$, $\ell_q^\mu \to \ell_2^\mu$ are continuous.*

Proof. See [6].

The Hankel operator $H[\phi]$ is represented by the matrix (ϕ_{i+j+1}), $i, j \geq 0$, and enjoys the following properties.

Theorem 4 *Let $p > 2$, $-1/p < \mu < 1/q$. Suppose ϕ, ψ are of the form (19), have no common discontinuities and that there exists a smooth partition of unity f, g such that ϕf and ψg have Fourier coefficients a_n satisfying $\sum_{-\infty}^{\infty} |a_n| n^3 < \infty$. Let $\tilde{\psi}(\theta) = \psi(-\theta)$. Then $H[\phi]H[\tilde{\psi}]$ can be realized as a sum of bounded operators which are compositions of the form*

$$AB : \ell_2^{\mu-1/2+1/p-\epsilon} \to \ell_p^{\mu} \tag{21}$$

where $B : \ell_2^{\mu-1/2+1/p-\epsilon} \to \ell_2^{\mu}$, B is trace class, and $A : \ell_2^{\mu} \to \ell_p^{\mu}$ for some sufficiently small ϵ.

Proof. Consider the identity found in [1]

$$H[\phi]H[\tilde{\psi}] = H[\phi f]H[\tilde{\psi}] + H[\phi]H[\tilde{g\psi}] + H[\phi]H[\tilde{f}]T[\phi] + H[\phi]H[\tilde{g}]T[\psi]. \tag{22}$$

We will show that the operator $H[h]$ is trace class from $\ell_2^{\mu-1/2+1/p-\epsilon}$ to ℓ_2^{μ}, where h is one of the functions ϕf, $\tilde{f}$, $\tilde{g}$, or $\tilde{g\psi}$. The operators $H[h]$ can be factored as CD where

$$D : \ell_2^{\mu-1/2+1/p-\epsilon} \to \ell_2^{\mu},\ D_{jk} = h_{j+k+1}(j+1) \tag{23}$$

$$C : \ell_2^{\mu} \to \ell_2^{\mu},\ C_{jk} = \frac{\delta_{jk}}{j+1} \tag{24}$$

are both Hilbert-Schmidt, as is easily seen by noting that

$$\sum_{j,k} |a_{j+k+1}|^2 (j+1)^{2+2\mu}(k+1)^{-2\mu+1-2/p+2\epsilon} < \infty. \tag{25}$$

Choosing ϵ sufficiently small, the operators $H[\tilde{\psi}]$, $T[\psi]$, and $T[\phi]$ are bounded on $\ell_2^{\mu-1/2+1/p-\epsilon}$. Finally, the operators $H[\phi f]$ and $H[\phi]$ are bounded from ℓ_2^{μ} to ℓ_p^{μ} since the inclusion is bounded from ℓ_2^{μ} to ℓ_p^{μ} and the Hankel operators are bounded on ℓ_p^{μ}.

We now show how the conjecture holds in the case $-1 < \mathrm{Re}\,\beta_r < 0$ and b sufficiently nice. The proof follows exactly that found in [1] or [4] with the appropriate modifications of the spaces. The idea of using cleverly chosen spaces goes back to Böttcher and Silbermann in [6]. The method used in the previous papers hinges on the fundamental identity

$$T_n[\phi\psi] = T_n[\phi]T_n[\psi] + P_n T[\phi] Q_n T[\psi] P_n + P_n H[\phi] H[\tilde{\psi}] P_n. \tag{26}$$

Here P_n is the projection defined on any ℓ_p^{μ} as

$$P_n(a_0, a_1, \ldots, a_{n-1}, a_n, \ldots) = (a_0, a_1, \ldots, a_{n-1}, 0, 0, \ldots)$$

and $Q_n = I - P_n$. Formula (26) above can be rewritten as

$$\begin{aligned} T_n[\psi]^{-1}T_n[\phi]^{-1}T_n[\phi\psi] &= I_n + A_n + B_n & (27)\\ &= (I_n + B_n)(I_n + A_n) - B_n A_n & (28)\\ A_n &= T_n[\psi]^{-1}T_n[\phi]^{-1}P_n H[\phi]H[\tilde{\psi}]P_n & (29)\\ B_n &= T_n[\psi]^{-1}T_n[\phi]^{-1}P_n T[\phi]Q_n T[\psi]P_n. & (30)\end{aligned}$$

We now assume $\phi(\theta)$ has the form given in (19) such that $b(\theta)$ does not vanish and has index zero. Suppose $b \in W_{1,1}^{3,3}$ and $-1 < \operatorname{Re}\beta_r < 0$. Then we can first find $p > 2$ and μ so that Theorem 2 holds. Consider first the matrix A_n as an operator on $\ell_2^{\mu-1/2+1/p-\epsilon}$ by seeing it as the composition of the following operators:

$$\begin{aligned} P_n H[\phi]H[\tilde{\psi}]P_n &: \ell_2^{\mu-1/2+1/p-\epsilon} \to \ell_p^{\mu}\\ T_n[\psi]^{-1}T_n[\phi]^{-1} &: \ell_p^{\mu} \to \ell_p^{\mu}\\ i &: \ell_p^{\mu} \to \ell_2^{\mu-1/2+1/p-\epsilon}\end{aligned}$$

From Theorem 4, the operator $H[\phi]H[\tilde{\psi}]$ is trace class with the appropriate choice of ϵ. Thus, the sequence of operators A_n converges in the trace norm. Since B_n converges to zero strongly, the product $B_n A_n$ will also converge to zero in the trace norm. Thus, just as in [1] or [4]

$$\det T_n[\phi]^{-1}T_n[\psi]^{-1}T_n[\phi\psi] \sim \det(I_n + A_n)\det(I_n + B_n). \tag{31}$$

This last statement yields the localization needed to prove (9) by induction on the number of singularities; since A_n converges in the trace norm to an operator A on $\ell_2^{\mu-1/2+1/p-\epsilon}$ and $\det(I_n + B_n) = \det(I_n + B_n')$, where, again as in [1], B_n' converges to an operator B' which is trace class on $\ell_2^{-(\mu-1/2+1/p-\epsilon)}$. To summarize, we have

Theorem 5 *Suppose* $\phi(\theta) = b(\theta)\prod_{r=1}^{R} t_{\beta_r}(\theta - \theta_r)$ *where*

(i) $b(\theta)$ *does not vanish and has index zero,*

(ii) $-1 < \operatorname{Re}\beta_r \le 0$,

(iii) $b \in W_{1,1}^{3,3}$.

Then (9) holds.

The above localization shows that the asymptotic formula in (9) holds. The fact that the constants agree follows from the validity of the conjecture for $|\mathrm{Re}\,\beta_r| < 1/2$ and the fact that the constants must be analytic functions of β_r. Also, note that by taking conjugates the conjecture also holds if $0 < \mathrm{Re}\,\beta_r < 1$.

§3. Some Special Cases

Let us consider $\phi(\theta) = t_{\beta_1}(\theta)t_{\beta_2}(\theta+\pi)$ with any arbitrary β_1 and β_2. A simple calculation shows there is more than one contributing representation in (9) if and only if $\mathrm{Re}\,\beta_2 - \mathrm{Re}\,\beta_1$ equals some odd integer. To see this, first note that if we pick one representation in (5), say using β_1 and β_2, then any other must be of the form $\beta_1 + k$ and $\beta_2 - k$ where k is some integer. Let $\beta_1 = u_1 + iv_1$ and $\beta_2 = u_2 + iv_2$. Then

$$\mathrm{Re}\,\Omega(k) = -2k^2 - 2u_1k + 2u_2k - {u_1}^2 + {v_1}^2 - {u_2}^2 + {v_2}^2.$$

This quadratic function of k has exactly two maxima for integer values of k when $u_1 - u_2$ is some odd integer. Conversely, if $u_1 - u_2 = 2j+1$, then the pairs $\beta_1 + j$, $\beta_2 - j$ and $\beta_1 + j + 1$, $\beta_2 - j + 1$ yield two maximizing representations. If $\mathrm{Re}\,\beta_2 - \mathrm{Re}\,\beta_1 = 2j+1$ for an integer j, then the two maximizing representations are, up to a sign, given by

$$\phi(\theta) = t_{(\beta_1+j)}(\theta)t_{(\beta_2-j)}(\theta+\pi) \tag{32}$$

and

$$\phi(\theta) = (-1)t_{(\beta_1+j+1)}(\theta)t_{(\beta_2-j-1)}(\theta+\pi) \tag{33}$$

Let us assume that we have picked one of those representations as a starting point, i.e. that the β_r's differ by one. Then for this new choice $\mathrm{Re}\,\beta_2 - \mathrm{Re}\,\beta_1 = 1$.

The prediction from the Extended Fisher-Hartwig Conjecture is that

$$\begin{aligned} D_n[\phi] \quad \sim \quad & n^{-{\beta_1}^2-{\beta_2}^2} 2^{2\beta_1\beta_2} G(1+\beta_1)G(1-\beta_1)G(1+\beta_2)G(1-\beta_2) \\ & \times \left(1 + (-1)^n (2n)^{2(\mathrm{Im}\,\beta_2 - \mathrm{Im}\,\beta_1)} \frac{\Gamma(1+\beta_1)\Gamma(1-\beta_2)}{\Gamma(-\beta_1)\Gamma(\beta_2)}\right) \end{aligned} \tag{34}$$

We can also easily compute the Fourier coefficients for this function.

$$\phi_n = \begin{cases} \frac{1}{\pi(\beta_2+\beta_1-n)}(\sin\beta_2\pi + \sin\beta_1\pi) & \text{if } n \text{ is even} \\ \frac{1}{\pi(\beta_2+\beta_1-n)}(\sin\beta_1\pi - \sin\beta_2\pi) & \text{if } n \text{ is odd} \end{cases} \tag{35}$$

Thus, if β_1 and β_2 satisfy $\beta_2 = \beta_1 + 1$, we have

$$\phi_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{2\sin\beta_1\pi}{\pi(\beta_2+\beta_1-n)} & \text{if } n \text{ is odd} \end{cases} \tag{36}$$

Now by rearranging the rows and columnms in an obvious way so that even and odd coefficients occur in blocks one can easily see that $D_n[\phi] = 0$ for n odd. For n even, this same arrangement yields

$$D_n[\phi] = D_{n/2}[t_{\gamma_1}]D_{n/2}[t_{\gamma_2}] \tag{37}$$

where $\gamma_1 = \frac{\beta_1+\beta_2+1}{2}$ and $\gamma_2 = \frac{\beta_1+\beta_2-1}{2}$, from which it follows that $\gamma_1 = \beta_1 + 1$ and $\gamma_2 = \beta_1$. A routine computation shows that this agrees with (9). Thus we have proved (9) in the case that $\beta_2 = \beta_1 + 2j + 1$ and $\theta_2 = \theta_1 + \pi$.

We now turn to some examples which illustrate the implication of the Extended Fisher-Hartwig Conjecture for the distribution of eigenvalues of Toeplitz matrices. Consider the function

$$\phi(\theta) = \begin{cases} e^{i(3\pi/4-\theta)} & \text{if } 0 < \theta < \pi \\ e^{i(-3\pi/4-\theta)} & \text{if } \pi < \theta < 2\pi \end{cases} \tag{38}$$

If we look at the equation $\det(T_n[\phi] - \lambda I_n) = 0$, we see that the values of the β_r's in the canonical product for $\phi - \lambda$ will be given by the correct choice of the parameters

$$\begin{aligned} \beta_1(\lambda) &= \frac{1}{2\pi i}\log\Big(\frac{\phi(0^-) - \lambda}{\phi(0^+) - \lambda}\Big) \\ \beta_2(\lambda) &= \frac{1}{2\pi i}\log\Big(\frac{\phi(\pi^-) - \lambda}{\phi(\pi^+) - \lambda}\Big). \end{aligned} \tag{39}$$

A quick computation of the arguments shows that for all functions of the form $\phi - \lambda$ with λ not in the image, either Theorem 5 applies or that $-1/2 < \operatorname{Re}\beta_r < 1/2$, in which case earlier results can be used. Since the determinant does not vanish asymptotically, this implies that the eigenvalues will cluster around the image. The following figure shows that the numerical approximation of the eigenvalues for $T_{50}[\phi]$ nicely illustrates the results of section 2.

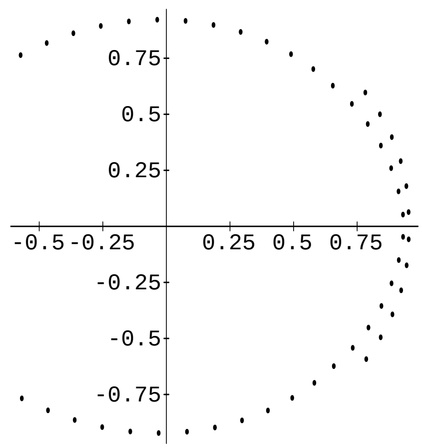

Figure 1. Eigenvalues of $T_{50}[\phi]$.

Another interesting example, described more fully in [2], is one with symbol

$$\phi(\theta) = \begin{cases} \theta + i & \text{if } 0 < \theta < \pi \\ \theta + 2i & \text{if } \pi < \theta < 2\pi \end{cases} \tag{40}$$

This function has two discontinuities and the range is two disjoint line segments. The extended conjecture would imply, as was pointed out earlier, that $\det T_n[\phi - \lambda] \neq 0$, for large n, except when λ is in the image of ϕ or when $-\mathrm{Re}\,\beta_1(\lambda) + \mathrm{Re}\,\beta_2(\lambda) = l$, where l is odd. The real parts of $\beta_1(\lambda)$ and $\beta_2(\lambda)$ are given by an appropriate choice of arguments for the logarithms found in formula (39). Let $\lambda = x + iy$ and solve for the arguments. Then take the tangents of both sides of the equation

$$-\mathrm{Re}\,\beta_1(\lambda) + \mathrm{Re}\,\beta_2(\lambda) = l$$

to arrive at this cubic equation that x and y must satisfy:

$$2y^3 + 2yx^2 - 4\pi xy - 9y^2 - 3x^2 + 5\pi x + (13 + 2\pi^2)y - 2\pi^2 - 6 = 0. \tag{41}$$

The parameter l disappears here since $\tan 2\pi l = 0$.

The next two pictures show that there are some "stray" eigenvalues that do not lie close to the image.

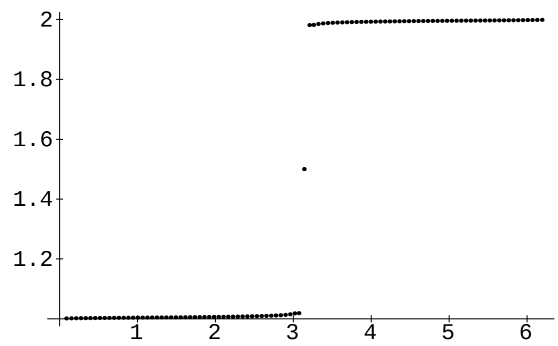

Figure 2. Eigenvalues of $T_{101}[\phi]$.

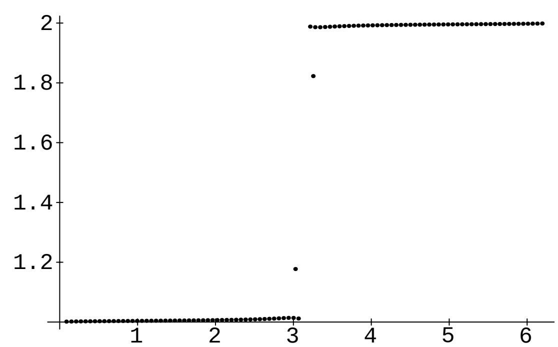

Figure 3. Eigenvalues of $T_{102}[\phi]$.

The next diagram shows a plot of the cubic curve in formula (41). Notice that the stray eigenvalues in the previous plots are near this cubic curve.

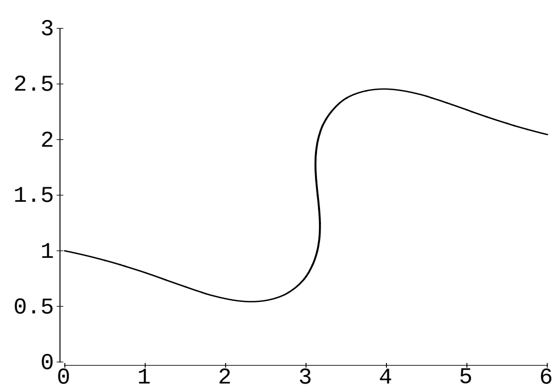

Figure 4. Eigenvalues lie near this cubic curve.

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